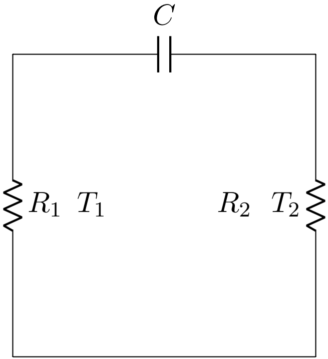
A nonequilibrium thermodynamic system:



Non-equilibrium thermodynamics of linear electrical circuits

Nahuel Freitas, Jean-Charles Delvenne (speak.), Massimiliano Esposito

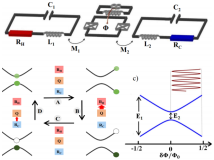
UCLouvain, University of Luxembourg

5 July 2019

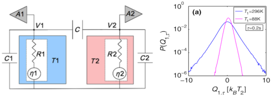
arXiv:1906.11233

Motivation and context

- Electronic circuits are a versatile framework to experimentally explore classical and quantum non-equilibrium thermodynamics.



Karimi, B., et al., PRB 94.18 (2016): 184503



Ciliberto, S., et al. PRL 110.18 (2013):180601

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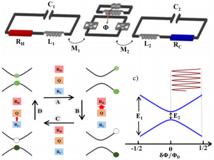

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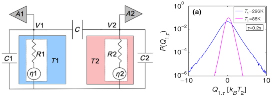

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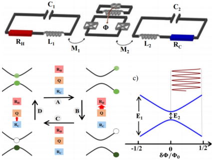

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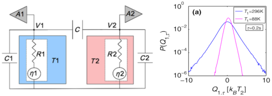

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- On the other hand, the main limitation in current technologies for information processing is given by the power consumption and by the generation of heat.
- Can the new developments in stochastic and quantum thermodynamics assist in the search for new strategies to reduce heat production within electronic circuits, or to improve its management?

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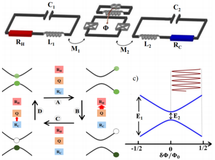

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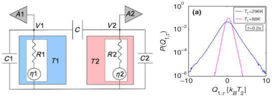

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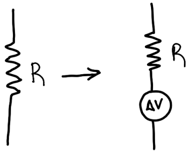
- On the other hand, the main limitation in current technologies for information processing is given by the power consumption and by the generation of heat.
- Can the new developments in stochastic and quantum thermodynamics assist in the search for new strategies to reduce heat production within electronic circuits, or to improve its management?

We want to build a general thermodynamic theory of electronic circuits.
First step: linear circuits.

Stochastic dynamics

High temperature Johnson-Nyquist noise:



- $\langle \Delta v(t) \rangle = 0$
- $\langle \Delta v(t) \Delta v(t') \rangle = 2 R k_b T \, \delta(t - t')$

Langevin state equation (=port-Hamiltonian + noise)

$$\frac{dx}{dt} = \mathcal{A}(t)\mathcal{H}(t)\, x + \mathcal{B}(t) s(t) + \sum_r \sqrt{2 k_b T_r}\, \mathcal{C}_r \, \xi(t)$$

$$\langle \xi_i(t) \xi_j(t') \rangle = \delta_{i,j} \delta(t - t')$$

Fluctuation-dissipation relation:

$$(\mathcal{A})_s = \frac{\mathcal{A} + \mathcal{A}^T}{2} = -\sum_r \mathcal{C}_r \mathcal{C}_r^T$$

Stochastic dynamics

Evolution of the mean values and covariance matrix

Mean values evolve with the deterministic equation of motion:

$$\frac{d\langle x\rangle}{dt} = \mathcal{A}\mathcal{H}(t)\,\langle x\rangle + \mathcal{B}(t)s(t)$$

And the covariance matrix $\sigma = \langle xx^T\rangle - \langle x\rangle\langle x\rangle^T$ evolves acording to:

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r\,\mathcal{C}_r\mathcal{C}_r^T$$

For time independent systems the stationary state covariance matrix can be obtained by solving a Lyapunov equation:

$$0 = \mathcal{A}\mathcal{H}\sigma_{\mathsf{st}} + \sigma_{\mathsf{st}}\mathcal{H}\mathcal{A}^T + \sum_r 2k_b T_r\,\mathcal{C}_r\mathcal{C}_r^T$$

Stochastic thermodynamics

Definition of local heat currents

Circuit energy and its variation:

$$E = \frac{1}{2} x^T \mathcal{H}(t) x \qquad \Longrightarrow \qquad \langle E \rangle = \frac{1}{2} \operatorname{Tr}\left[\mathcal{H}(t) \langle x \rangle \langle x \rangle^T\right] + \frac{1}{2} \operatorname{Tr}\left[\mathcal{H}\sigma\right]$$

$$\frac{d\langle E \rangle}{dt} = \underbrace{\frac{1}{2} \operatorname{Tr}\left[\mathcal{H}(t) \frac{d}{dt}\left(\langle x \rangle \langle x \rangle^T + \sigma\right)\right]}_{\text{Heat}} + \underbrace{\frac{1}{2} \operatorname{Tr}\left[\frac{d}{dt}\mathcal{H}(t)\left(\langle x \rangle \langle x \rangle^T + \sigma\right)\right]}_{\text{Work}}$$

Employing the evolution equation for σ and the FD relation, we obtain:

$$\langle \dot{Q} \rangle = \sum_r \left(\langle i_r \rangle \langle v_r \rangle + \operatorname{Tr}[(\mathcal{H}\sigma\mathcal{H} - k_b T_r \mathcal{H}) \mathcal{C}_r \mathcal{C}_r^T] \right),$$

From this it is tempting to define the local heat currents as:

$$\langle \dot{Q}_r \rangle = \langle i_r \rangle \langle v_r \rangle + \operatorname{Tr}[(\mathcal{H}\sigma\mathcal{H} - k_b T_r \mathcal{H}) \mathcal{C}_r \mathcal{C}_r^T].$$

However, THIS IS NOT ALWAYS CORRECT!

Stochastic thermodynamics

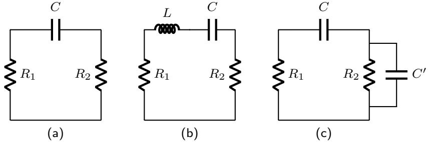
Definition of local heat currents

Local heat currents can be naturally defined as:

$$\dot{Q}_r = i_r(v_r + \Delta v_r)$$

However, $\langle \dot{Q}_r \rangle$ is divergent in general. Why?

Some examples:



In (a), fluctuations of arbitrarily high frequency in R_2 can be dissipated into R_1.
In (b) and (c) these fluctuations are filtered out.

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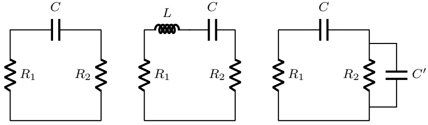

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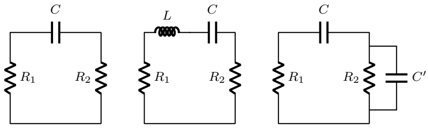

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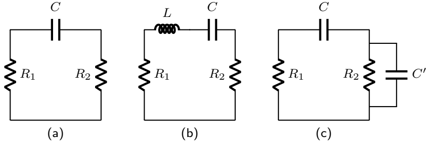

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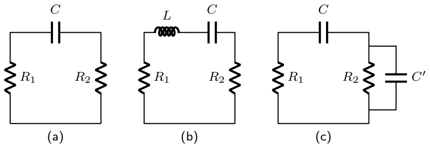

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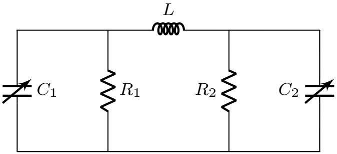
- However, it indicates whether stray but relevant degrees of freedom are not explicitly described.
- We can also replace white noise with Planck's distribution
- General topological condition to check if heat currents are well defined

Application: cooling a resistor

A simple circuit-based machine



The state equation is:

$$\frac{dx}{dt} = \mathcal{A}(t)\mathcal{H}(t)\,x + \sqrt{2k_bT_1}\,\mathcal{C}_1\,\xi(t) + \sqrt{2k_bT_2}\,\mathcal{C}_2\,\xi(t)$$

$$\mathcal{A}(t) = \left[\begin{array}{cc|c} -R_1^{-1} & 0 & 1 \\ 0 & -R_2^{-1} & 1 \\ \hline -1 & -1 & 0 \end{array}\right]$$

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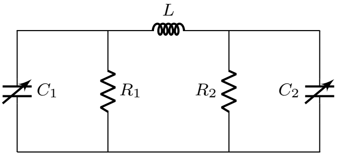

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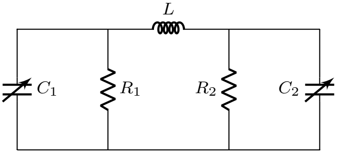
$$\mathcal{H}(t) = \begin{bmatrix} C_1(t) & & \\ & C_2(t) & \\ & & L \end{bmatrix}$$

Application: cooling a resistor

A simple circuit-based machine



1 - Stationary heat conduction: $(C_1 = C_2 = C,\ R_1 = R_2 = R)$

$$0 = \mathcal{A}\mathcal{H}\sigma_{\mathsf{st}} + \sigma_{\mathsf{st}}\mathcal{H}\mathcal{A}^T + \sum_r 2k_b T_r\, \mathcal{C}_r\mathcal{C}_r^T$$

$$\sigma_{\mathsf{st}} = k_b\frac{T_1+T_2}{2}\mathcal{H}^{-1} + \frac{k_b(T_2-T_1)}{2}\frac{CL}{CR^2+L}\begin{bmatrix} 1 & 0 & -R \\ 0 & -1 & R \\ -R & R & 0 \end{bmatrix}$$

$$\langle\dot{Q}_r\rangle = \mathrm{Tr}[(\mathcal{H}\sigma\mathcal{H} - k_bT_r\mathcal{H})\mathcal{C}_r\mathcal{C}_r^T] \implies \langle\dot{Q}_1\rangle = -\langle\dot{Q}_2\rangle = -\frac{k_b(T_2-T_1}{2}\frac{R}{CR^2+L}$$

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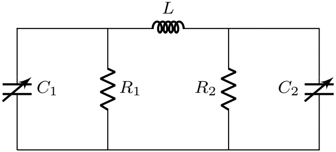


2 - Isothermal refrigeration ($T_1 = T_2 = T$)

We consider a simple driving protocol:

$$C_1 = C + \Delta C \cos(\omega_d t)$$
$$C_2 = C + \Delta C \cos(\omega_d t + \phi)$$

And compute the asymptotic cycle averages:

$$\langle \dot{X} \rangle_c = \lim_{t\to\infty} \frac{\omega_d}{2\pi} \int_t^{t+\frac{2\pi}{\omega_d}} \langle \dot{X} \rangle$$

In this case, we need to solve:

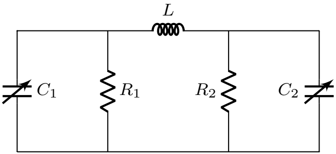
$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r \, \mathcal{C}_r \mathcal{C}_r^T$$

For periodic driving, under stability conditions, $\sigma(t)$ is asymptotically periodic:

$$\mathcal{H}(t) = \sum_{k=-\infty}^{+\infty} \mathcal{H}_k \, e^{ik\omega_d t}. \quad \underset{\text{asymptotically}}{\Longrightarrow} \quad \sigma(t) = \sum_{k,k'=-\infty}^{+\infty} \sigma_{k,k'} \, e^{i(k-k')\omega_d t}$$

Application: cooling a resistor

A simple circuit-based machine



3 - Isothermal refrigeration ($T_1 = T_2 = T$)

In the weak ($\Delta C \ll C$) and slow ($\omega_d \ll 1/\sqrt{LC}$) driving limit we can obtain analytical results:

$$\langle \dot{W} \rangle_c = \langle \dot{Q}_1 \rangle_c + \langle \dot{Q}_2 \rangle_c = k_b T \, (\omega_d \Delta C)^2 \, \frac{R(CR^2 \cos(\theta) + CR^2 + 2L)}{8C(CR^2 + L)} + \mathcal{O}(\omega_d^3)$$

$$\langle \dot{Q}_{1/2} \rangle_c = \mp k_b T \, \omega_d (\Delta C)^2 \frac{R^4 \sin(\theta)}{8(CR^2 + L)^2} + \frac{\langle \dot{W} \rangle_c}{2} + \mathcal{O}(\omega_d^3)$$

Coefficient of Perfomance:

$$\text{CoP} = \frac{|\langle \dot{Q}_1 \rangle_c|}{\langle \dot{W} \rangle_c} = \frac{1}{\omega_d} \frac{R \sin(\theta)/(CR^2 + L)}{\cos(\theta) + 1 + 2L/(CR^2)} - \frac{1}{2}$$

Optimal driving frequency:

$$\omega_d^{\text{max}} = \frac{2(RC)^{-1}}{1 + L/(CR^2)} \frac{\sin(\theta)}{\cos(\theta) + 1 + 2L/(CR^2)}$$

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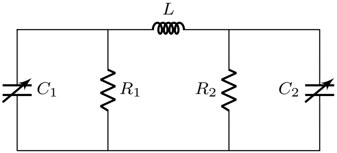
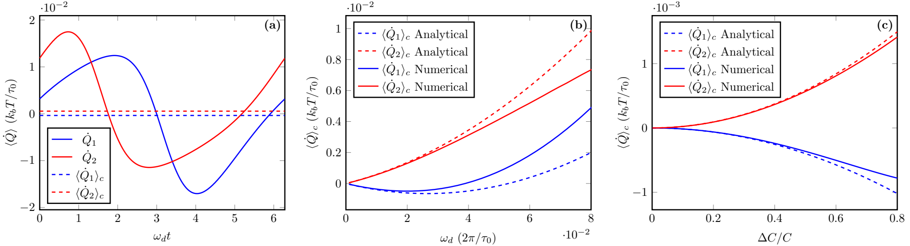


Numerical vs analytical results: ($\tau_0 = \sqrt{LC}$, $\tau_d = RC$, $\tau_0 = \tau_d$)



(a) Asymptotic cycle of the heat currents for $\Delta C/C = 1/2$ and $\omega_d/(2\pi) = 10^{-2}/\tau_d$ (dashed lines indicate cycle averages).
(b) Average heat currents versus driving frequency for $\Delta C/C = 0.5$.
(c) Average heat currents versus driving strength for $\omega_d/(2\pi) = 10^{-2}/\tau_d$.
For all cases we took $\theta = \pi/2$ and $T_1 = T_2 = T$.

How were these results obtained?

Generalized Lyapunov equation

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r\, \mathcal{C}_r \mathcal{C}_r^T$$

$$\mathcal{H}(t) = \sum_{k=-\infty}^{+\infty} \mathcal{H}_k\, e^{ik\omega_d t}. \qquad \underset{\text{asymptotically}}{\Longrightarrow} \qquad \sigma(t) = \sum_{k,k'=-\infty}^{+\infty} \sigma_{k,k'}\, e^{i(k-k')\omega_d t}$$

How to find the $\sigma_{k,k'}$ given $\mathcal{H}_k$? We can define:

$$S = \begin{bmatrix} \ddots & & & & \\ & \sigma^2_{-1,-1} & \sigma^2_{-1,0} & \sigma^2_{-1,1} & \\ & \sigma^2_{0,-1} & \sigma^2_{0,0} & \sigma^2_{0,1} & \\ & \sigma^2_{1,-1} & \sigma^2_{1,0} & \sigma^2_{1,1} & \\ & & & & \ddots \end{bmatrix} \qquad D_r = \begin{bmatrix} \ddots & & & & \\ & 0 & 0 & 0 & \\ & 0 & \mathcal{C}_r\mathcal{C}_r^T & 0 & \\ & 0 & 0 & 0 & \\ & & & & \ddots \end{bmatrix}$$

And then just solve the Lyapunov equation:

$$AS + SA^\dagger + \sum_r 2k_b T_r D_r = 0$$

Generalization to quantum noise

Preliminary comments

Canonical quantization of electrical circuits:

1. Write down a Lagrangian for the circuit $\mathcal{L}(x, \dot{x})$

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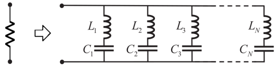
3. Promote x and p to quantum mechanical operators and impose canonical commutation relations $[x_i, p_j] = i\hbar\delta_{i,j}$
4. If there are resistors, we can explicitly model them as collections of harmonic modes (Caldeira-Legget model)



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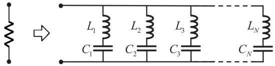
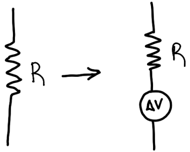


5. In the end, we obtain:

$$H_T = H_{\text{circuit}} + \sum_r H_r + \sum_r H_{\text{int},r}$$

Generalization to quantum noise

Semiclassical treatment



Classical Johnson-Nyquist noise:

$$\langle \Delta v(t)\Delta v(t')\rangle = 2Rk_bT\,\delta(t-t') \implies S(\omega) = \frac{Rk_bT}{\pi}$$

Quantum Johnshon-Nyquist noise:

$$`\langle \Delta v(t)\Delta v(t')\rangle' = \left\langle \frac{\Delta v(t)\Delta v(t') + \Delta v(t')\Delta v(t)}{2} \right\rangle = f(t,t')$$

$$\mathcal{S}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\tau\, e^{-i\omega\tau} f(t,t') = \frac{R}{\pi}\hbar\omega\ \coth\left(\frac{\hbar\omega}{2k_bT}\right)$$

$$= \frac{R}{2\pi}\hbar\omega\,(N(\omega)+1/2)$$

Generalization to quantum noise

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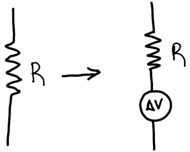

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Semiclassical treatment:

$$\frac{dx}{dt} = \mathcal{A}(t)\mathcal{H}(t)\,x + \mathcal{B}(t)s(t) + \sum_r \sqrt{2k_bT_r}\,\mathcal{C}_r\,\xi(t), \qquad \mathcal{S}_{\xi_r}(\omega) = \frac{1}{2\pi}\frac{\hbar\omega}{k_bT_r}\,(N_r(\omega)+1/2)$$

This "semi-classical" treatment is correct, equivalent to full quantum treatment!

Generalization to quantum noise

Green's function techniques

For $y = x - \langle x \rangle$

$$\frac{dy}{dt} = \mathcal{A}(t)\mathcal{H}(t)\, y + \sum_r \sqrt{2k_b T_r}\, \mathcal{C}_r\, \xi(t), \qquad \mathcal{S}_{\xi_r}(\omega) = \frac{1}{2\pi} \frac{\hbar\omega}{k_b T_r} \left(N_r(\omega) + 1/2\right)$$

We introduce the Green's function of the circuit:

$$\frac{d}{dt} G(t,t') - \mathcal{A}(t)\mathcal{H}(t) G(t,t') = \mathbb{1}\delta(t,t')$$

And we have the formal solution:

$$y(t) = G(t,0)\, y(0) + \int_0^t d\tau\, G(t,\tau) \sum_r \sqrt{2k_b T_r}\, \mathcal{C}_r(\tau)\, \xi(\tau)$$

Using this, we can find that:

$$\begin{aligned}
\sigma(t) &= \langle y(t) y(t)^T \rangle \\
&= G(t,0)\sigma(0)G(t,0)^T + \\
&\int_0^t d\tau \sum_r \sqrt{2k_b T_r} \left[G(t,0)\langle y(0)\xi^T(\tau)\rangle \mathcal{C}_r(\tau)^T G(t,\tau)^T + G(t,\tau)\mathcal{C}_r(\tau)\langle \xi(\tau) y(0)^T\rangle G(t,0)^T \right] + \\
&\int_0^t d\tau \int_0^t d\tau' \sum_{r,r'} 2k_b \sqrt{T_r T_{r'}}\, G(t,\tau)\mathcal{C}_r(\tau)\langle \xi(\tau)\xi(\tau')^T\rangle \mathcal{C}_{r'}(\tau')^T G(t,\tau')^T.
\end{aligned}$$

Generalization to quantum noise

Green's function techniques

Differential equation for the covariance matrix:

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r \left(\mathcal{I}_r(t)\,\mathcal{C}_r\mathcal{C}_r^T + \mathcal{C}_r\mathcal{C}_r^T\,\mathcal{I}_r(t)^T\right)$$

where:

$$\mathcal{I}_r(t) = \int_0^t d\tau\, G(t, t-\tau)\,\langle \xi_r(0)\xi_r(\tau)\rangle$$

Classical limit:

In the limit of large temperatures we have $\langle \xi_r(0)\xi_r(\tau)\rangle \to \delta(t-t')$ and therefore

$$\mathcal{I}_r(t) = \int_0^t d\tau\, G(t, t-\tau)\,\langle \xi_r(0)\xi_r(\tau)\rangle \to \frac{1}{2}G(t,t) = \frac{\mathbb{1}}{2}$$

and we recover the classical result:

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r\,\mathcal{C}_r\mathcal{C}_r^T$$

Generalization to quantum noise

Quantum heat currents

Differential equation for the covariance matrix:

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r \left(\mathcal{I}_r(t)\,\mathcal{C}_r\mathcal{C}_r^T + \mathcal{C}_r\mathcal{C}_r^T\,\mathcal{I}_r(t)^T\right)$$

Quantum heat currents:

$$\langle\dot{Q}\rangle = \frac{1}{2}\,\mathrm{Tr}\left[\mathcal{H}(t)\frac{d}{dt}\left(\langle x\rangle\langle x\rangle^T + \sigma\right)\right]$$

Again, under the condition $Q_{\mathsf{RR}} = 0$, we have:

$$\langle\dot{Q}_r\rangle = \langle i_r\rangle\langle v_r\rangle + \mathrm{Tr}\left[(\mathcal{H}\sigma(t)\mathcal{H} - 2k_b T_r\mathcal{H}\mathcal{I}_r(t))\mathcal{C}_r\mathcal{C}_r^T\right]$$

Quantum heat currents in frequency domain

Partial transform of the Green's function

$$\hat{G}(t,\omega) = \int_0^t d\tau \, e^{-i\omega(t-\tau)} \, G(t,\tau) \qquad \Longrightarrow \qquad \frac{d}{dt}\hat{G}(t,\omega) = \mathbb{1} - [i\omega - \mathcal{A}\mathcal{H}(t)]\hat{G}(t,\omega)$$

Important property: $\frac{d}{dt}\Big(\hat{G}^\dagger \mathcal{H} \hat{G}\Big) - \hat{G}^\dagger \frac{d\mathcal{H}}{dt} \hat{G} - 2\hat{G}^\dagger \mathcal{H}(\mathcal{A})_s \mathcal{H}\hat{G} = \mathcal{H}\hat{G} + \hat{G}^\dagger \mathcal{H}$

Convolution function and covariance matrix

$$\mathcal{I}_r(t) = \frac{1}{2\pi k_b T_r} \int_{-\Lambda}^{+\Lambda} d\omega \, \hbar\omega \, \hat{G}(t,\omega)(N_r(\omega) + 1/2)$$

Λ: High frequency cut-off

$$\sigma(t) = \frac{1}{\pi} \sum_r \int_{-\Lambda}^{+\Lambda} d\omega \, \hbar\omega \, \hat{G}(t,\omega)\mathcal{D}_r \hat{G}(t,\omega)^\dagger (N_r(\omega) + 1/2)$$

$\mathcal{D}_r = \mathcal{C}_r \mathcal{C}_r^T$

We can now enter all this information in our expression for the local heat currents

$$\langle \dot{Q}_r \rangle = \langle i_r \rangle \langle v_r \rangle + \mathrm{Tr}\left[(\mathcal{H}\sigma(t)\mathcal{H} - 2k_b T_r \mathcal{H}\mathcal{I}_r(t))\mathcal{C}_r \mathcal{C}_r^T \right]$$

Quantum heat currents in frequency domain

Generalization of Landauer-Büttiker formula

$$\langle \dot{Q}_r \rangle = \langle i_r \rangle \langle v_r \rangle + \sum_{r'} \int_{-\Lambda}^{+\Lambda} d\omega \, \hbar\omega \, f_{r,r'}(t,\omega) \left(N_{r'}(\omega) + 1/2 \right)$$

Non-diagonal elements: $f_{r,r'}(t,\omega) = \frac{1}{\pi} \operatorname{Tr} \left[\mathcal{H}(t) \hat{G}(t,\omega) \mathcal{D}_{r'} \hat{G}(t,\omega)^\dagger \mathcal{H}(t) \mathcal{D}_r \right] \qquad (r \neq r')$

Sum over first index: $\bar{f}_{r'}(t,\omega) = \sum_r f_{r,r'}(t,\omega) = \frac{1}{2\pi} \operatorname{Tr} \left[\left(\hat{G}^\dagger \frac{d\mathcal{H}}{dt} \hat{G} - \frac{d}{dt} \left(G^\dagger \mathcal{H} \hat{G} \right) \right) \mathcal{D}_{r'} \right]$

<u>Particular case:</u> for static circuits ($\bar{f}'_r = 0$) we recover the usual Landauer-Büttiker formula

$$\langle \dot{Q}_r \rangle = \langle i_r \rangle \langle v_r \rangle + \sum_{r'} \int_{-\Lambda}^{+\Lambda} d\omega \, \hbar\omega \, f_{r,r'}(\omega) \left(N_{r'}(\omega) - N_r(\omega) \right)$$

Main result

We have derived a generalized Landauer-Büttiker formula which is valid for arbitrary circuits, with any number of resistors at arbitrary temperatures, and for arbitrary driving protocols.

Landauer-Büttiker formula for periodic driving

Again, we consider periodic parametric driving: $\mathcal{H}(t) = \sum_{k=-\infty}^{+\infty} \mathcal{H}_k \, e^{ik\omega_d t}$

Then $G(t,\omega)$ is asymptotically periodic: $\hat{G}(t,\omega) = \sum_{j=-\infty}^{+\infty} \hat{G}_j(\omega) \, e^{ij\omega_d t}$, where:

$$i(\omega + j\omega_d)\hat{G}_j(\omega) = \mathbb{1}\delta_{j,0} + \mathcal{A}\sum_k \mathcal{H}_k \hat{G}_{j-k}(\omega)$$

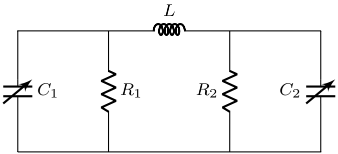
Then:

$$\langle \dot{Q}_r \rangle_c = \langle \langle i_r \rangle \langle v_r \rangle \rangle_c + \sum_{r'} \int_{-\Lambda}^{+\Lambda} d\omega \, \hbar\omega \, F_{r,r'}(\omega) \, (N_{r'}(\omega) + 1/2)$$

$$F_{r,r'}(\omega) = \frac{1}{\pi} \sum_{j,j',k} \mathrm{Tr}\left[\mathcal{H}_k \hat{G}_j(\omega) \mathcal{D}_{r'} \hat{G}^{\dagger}_{j'}(\omega) \mathcal{H}_{j'-j-k} \mathcal{D}_r\right] \text{ for } r' \neq r \tag{1}$$

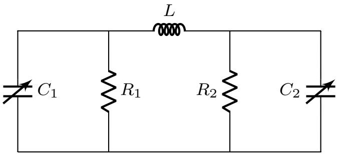
$$\bar{F}_{r'}(\omega) = \sum_r F_{r,r'}(\omega) = \frac{1}{2\pi} \sum_{j,k} ik\omega_d \, \mathrm{Tr}\left[\hat{G}^{\dagger}_j(\omega) \mathcal{H}_k \hat{G}_{j-k}(\omega) \mathcal{D}_{r'}\right] \tag{2}$$

Quantum limits for cooling



Going back to our cooling scheme, what happens if we enter the quantum regime?
Classical (dashed) vs quantum (solid) :

Quantum limits for cooling



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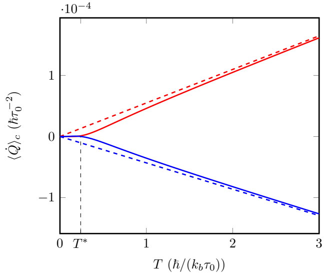

$\tau_0 = \sqrt{LC}$ and $\tau_d = RC$
Difficulty to cool down a cold system.

Summary and next steps

- We have considered the stochastic description of general RLC circuits

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