Hyperbolicity of the heat equation

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Outline

- Diffusive transport are generally described by Fourier's law:
  - Parabolic nature: Any initial disturbance in the material body is propagated instantly[1]
- Hyperbolicity: physically relevant and desired property $\rightarrow$ physically meaningful finite speed of phenomena propagation
- To eliminate this unphysical feature: modified Fourier law to take into account the thermal inertia $\rightarrow$ finite speed of propagation[2]

[1] Christo I. Christov and P. M. Jordan. "Heat conduction paradox involving second-sound propagation in moving media.". In: *Physical review letters* 94 15 (2005), pp. 154–301.

[2] C. Cattaneo. "On a form of heat equation which eliminates the paradox of instantaneous propagation". In: *C. R. Acad. Sci. Paris* (1958), pp. 431–433, P. Vernotte. "Les paradoxes de la theorie continue de l'equation de la chaleur". In: *C. R. Acad. Sci. Paris* 246 (1958), pp. 3154–3155, D. Jou, J. Casas-Vazquez, and G. Lebon. *Extended irreversible thermodynamics (3rd Edition)*. Springer, 2001.

- Application to a Stefan Problem is considered:
  - Specific type of boundary value problem (moving boundary position needs to be determined);
  - Generally focusing in the heat distribution, in a phase changing medium;
- Examples:
  - Diffusion of heat in the melting/solidification of ice;
  - Fluid flow in porous media;
  - Shock waves in gas dynamics[3].

[3] L.I. Rubenstein. *The Stefan problem, by L.I. Rubenstein*. Translations of mathematical monographs. American Mathematical Society, 1971.

- Considering a fluid at rest with constant density and neglecting non-linear terms in gradients and time-derivatives:

$$\rho C_p \frac{\partial T}{\partial t} = -\partial_\zeta q \tag{1}$$

- One dimension heat flux given by the Fourier's law:

$$q = -k\partial_\zeta T \tag{2}$$

- Gives the parabolic PDE and its solution for infinite domain:

$$\begin{gathered} \partial_t T = \alpha \partial_{\zeta\zeta} T \\ T(\zeta, t) = \frac{1}{(4\pi\alpha t)^{3/2}} \int_{-\infty}^{\infty} T(z, 0) \mathrm{e}^{\left(-\frac{(z-\zeta)^2}{4\alpha t}\right)} dz \end{gathered} \tag{3}$$

- If the initial condition is different from 0, this solution predicts a instant heat propagation → Heat Conduction Paradox.

- Cattaneo[4] modified the Fourier's law, arguing that there is a time-lag between the start of the particles at their point of departure and the time of passage through the middle layer:
  - if the temperature changes in time, the heat flux at a certain time depends on the temperature gradient at an earlier time
- This assumption leads to the following modified heat flux:

$$q = -k(1 - \tau\partial_t)\partial_\zeta T \tag{4}$$

- If τ is small:

$$(1 - \tau\partial_t)^{-1} \approx (1 + \tau\partial_t) \tag{5}$$

- The energy balance results in a second order hyperbolic PDE:

$$\tau\partial_{tt} T + \partial_t T = \alpha\partial_{\zeta\zeta} T \tag{6}$$

- If $\tau = 0$, the original parabolic PDE is obtained, as expected.

[4] C. Cattaneo. "On a form of heat equation which eliminates the paradox of instantaneous propagation". In: *C. R. Acad. Sci. Paris* (1958), pp. 431–433.

- Considering the hyperbolic and parabolic heat conduction equations with the following boundary conditions and initial condition, the results obtained for the same time interval are shown below:

$$\partial_\zeta T(\zeta = 0) = \partial_\zeta T(\zeta = 1) = 0$$
$$T_0(\zeta) = \begin{cases} 1 - \frac{(\zeta - 0.5)^2}{0.25^2}, \ 0.25 \le \zeta \le 0.75 \\ 0, \text{ otherwise} \end{cases} \tag{7}$$

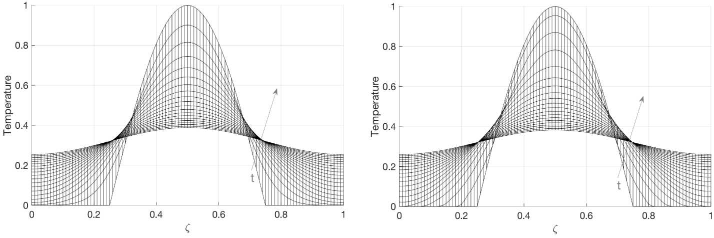


Figure 1: Comparison between the results from the Heat Equations: (Left) Parabolic; (Right) 2nd order Hyperbolic

Comparison in a one dimension heat diffusion problem

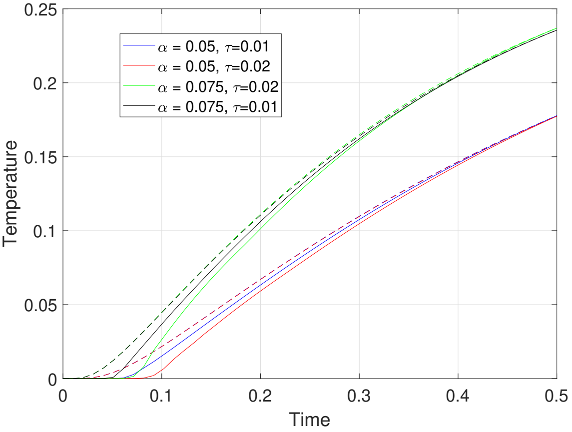


Figure 2: Comparison between the results from the Heat Equations for $\zeta = 0.1$: (Dashed) Parabolic; (Solid) 2nd order Hyperbolic

- The eigenvalue problem ($A\phi = \lambda\phi$, where A is the suitable operator) is solved for the hyperbolic and parabolic equations:
  - For the parabolic:
$$\lambda_n = -\alpha n^2 \pi^2 \tag{8}$$
  - Infinite set of unbounded real fast eigenvalues ($\lambda_n \to -\infty$ as $n \to \infty$)
  - For the hyperbolic:
$$\lambda_n = \frac{-1 \pm \sqrt{1 - 4\tau(\alpha n^2 \pi^2)}}{2\tau} \tag{9}$$
  - Real part of the eigenvalues is now bounded $\frac{-1}{\tau} < Re(\lambda_n) < 0$;
  - Eigenvalues with imaginary parts when $n > \sqrt{\frac{1}{4\tau\alpha\pi^2}}$;

- Considering the system of a phase change represented below:

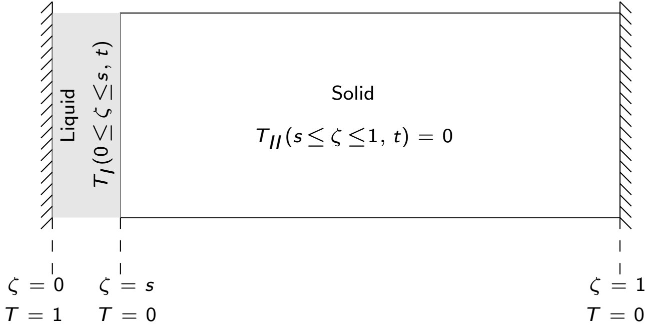


Figure 3: Representation of the system with phase transition

- Stefan condition:

$$C_l d_t s(t) = q_I(t, \zeta = s - \delta) - q_{II}(t, \zeta = s + \delta) \tag{10}$$

- The following assumptions are considered:
  - Transition temperature: $T_C = 0$;
  - Initial condition: $T_0(\zeta) = 0$;
  - Temperature at $\zeta = 1$ is the transition temperature ($T(\zeta = 1) = T_C$);
  - No heat sinks or sources;
- Simplify the model to a one phase problem (only the liquid phase has a temperature profile that needs to be found) and only this phase has a heat flux:

$$C_l d_t s(t) = q_l(t, \zeta = s - \delta); \tag{11}$$

- Considering the heat flux defined by the Fourier's and modified Fourier's law, the following systems are obtained:
  - Fourier's law (parabolic):

$$\begin{cases} d_t s = -\beta \, \partial_\zeta T_l \big|_{\zeta = s(t)} \\ \partial_t T_l = \alpha \partial_{\zeta\zeta} T_l, \; 0 \leq \zeta \leq s(t) \end{cases} \tag{12}$$

  - Modified Fourier's law (hyperbolic):

$$\begin{cases} \tau d_{tt} s + d_t s = -\beta \, \partial_\zeta T_l \big|_{\zeta = s(t)} \\ \tau \partial_{tt} T_l + \partial_t T_l = \alpha \partial_{\zeta\zeta} T_l, \; 0 \leq \zeta \leq s(t) \end{cases} \tag{13}$$

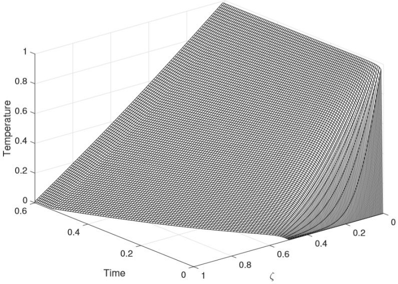
- A change of variables is applied in the spatial coordinate to change them to fixed domain problems: $\epsilon = \dfrac{\zeta}{s(t)} \rightarrow \epsilon \in [0, 1]$
- This transformation gives the following systems
  - Fourier's law (parabolic):

$$\begin{cases} d_t s = -\beta \dfrac{1}{s} \partial_\epsilon T_l(\epsilon, t) \\ \alpha \dfrac{1}{s^2} \partial_{\epsilon\epsilon} T_l(\epsilon, t) = \partial_t T_l(\epsilon, t) - \dfrac{\epsilon}{s} d_t s \partial_\epsilon T_l(\epsilon, t) \end{cases} \tag{14}$$

  - Modified Fourier's law (hyperbolic):

$$\begin{cases} \tau d_{tt} s + d_t s = -\beta \dfrac{1}{s} \partial_\epsilon T_l(\epsilon, t) \\ \alpha \dfrac{1}{s^2} \partial_{\epsilon\epsilon} T_l(\epsilon, t) = \tau \left\{ \partial_{tt} T_l(\epsilon, t) - \dfrac{2\epsilon}{s} d_t s \partial_\epsilon \partial_t T_l(\epsilon, t) + \right. \\ \left. \partial_\epsilon T_l(\epsilon, t) \left[\dfrac{2\epsilon}{s^2} (d_t s)^2 - \dfrac{\epsilon}{s} d_{tt} s \right] \right\} + \partial_t T_l(\epsilon, t) - \dfrac{\epsilon}{s} d_t s \partial_\epsilon T_l(\epsilon, t) \end{cases} \tag{15}$$

- The following figures show the results for the parabolic and hyperbolic PDEs:



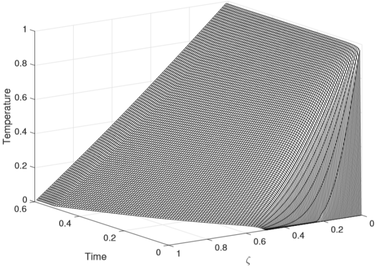


Figure 4: Stefan problem simulation results: (Left) Parabolic; (Right) 2nd order Hyperbolic

The Stefan Problem: Simulation results - Moving boundary

- The moving boundary dynamic:

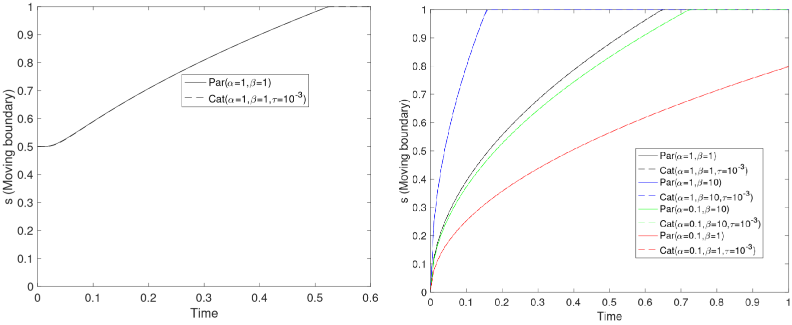


Figure 5: Stefan problem simulation results: (Left) Conditions from the previous results; (Right) Different parameters values

Conclusions

The analysis of the behavior of the heat diffusion considering hyperbolic and parabolic equations was accomplished. It was possible to see the difference between these two equations, which shows the finite speed of propagation of the hyperbolic equation.
A Stefan problem was also considered and the results show that the interface dynamic for the hyperbolic equation is similar to the parabolic **for the conditions considered**, but the former equation's finite speed of propagation would be more desirable to represent the dynamics of the actual physical system.

Future work will analyze the difference between these two equations when control problems are considered.

Thank You

References I

- C. Cattaneo. "On a form of heat equation which eliminates the paradox of instantaneous propagation". In: *C. R. Acad. Sci. Paris* (1958), pp. 431–433.
- Christo I. Christov and P. M. Jordan. "Heat conduction paradox involving second-sound propagation in moving media.". In: *Physical review letters* 94 15 (2005), pp. 154–301.
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