Efficient Analysis of Unambiguous Automata Using Matrix Semigroup Techniques

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Outline

1. Introduction
   - What is model checking
   - Finite word model checking
   - Infinite word model checking
2. Cuts
3. Pseudo-cuts

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- Calculating possibility of a scenario happening
- Example: “It’s always darkest before the dawn”
- Example: $\mathbf{G}(\mathit{darkest}(x) \rightarrow \mathbf{X}\mathit{dawn}(x))$
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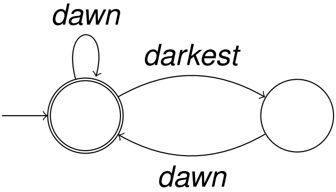

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- Probability distribution over sequences of events
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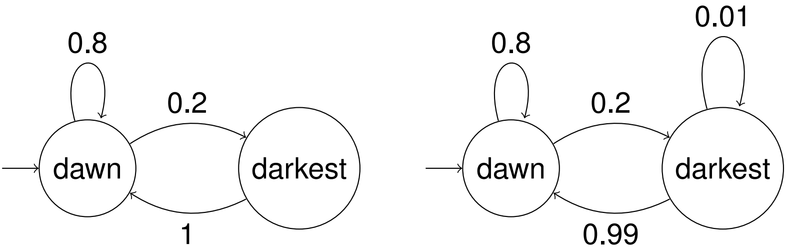

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DFA model checking

- Given a DFA (S, M, δ, q_0, F),
  a Markov chain (P, M), with $P_{m,m'}$ denoting the probability of going to $m' \in M$ from $m \in M$,
  an initial vector $i \in M$ stating the initial distribution of the Markov chain,
  and a vector $\omega \in [0, 1]^M$ denoting the probability of stopping the word after reading a character in M.
- Denote by $\zeta_{s,m}$ the probability of accepting a word starting with m in the DFA starting in s.
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- Notice that $\zeta_{q_0,m} = \omega(m) + (1-\omega(m)) \sum_{m' \in M} P_{m,m'} \zeta_{\delta(q_0,m),m'}$ if $\delta(q_0, m) \in F$ and $\zeta_{q_0,m} = (1-\omega(m)) \sum_{m' \in M} P_{m,m'} \zeta_{\delta(q_0,m),m'}$ otherwise.
- This leads to the following characterisation for ζ:

$$\zeta_{q,m} = \begin{cases} \omega(m) + (1-\omega(m)) \sum_{m' \in M} P_{m,m'} \zeta_{\delta(q,m),m'} & \text{if } \delta(q,m) \in F \\ (1-\omega(m)) \sum_{m' \in M} P_{m,m'} \zeta_{\delta(q,m),m'} & \text{otherwise.} \end{cases}$$

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- For UFAs:

$$\zeta_{q,m} = |\delta(q,m) \cap F| \omega(m) + (1-\omega(m)) \sum_{m' \in M} \sum_{q' \in \delta(q,m)} P_{m,m'} \zeta_{q',m'}$$

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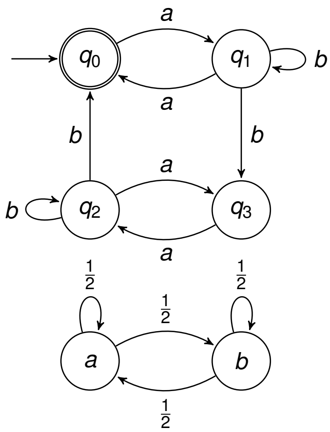

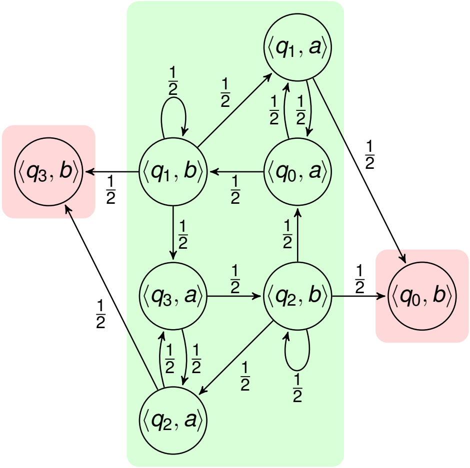

Cuts

- Notice that if the probability of generating an accepted word is greater than zero, then there exist prefixes such that almost every word beginning with that prefix is accepted
- Given such a prefix p for a UBA (S, M, δ, Q_0, F), we see that for any $m \in M$, $\sum_{q_0 \in Q_0} \sum_{s \in \delta(q_0, p)} \zeta_{s,m} \geq 1$. $\delta(q_0, p)$ is called a *cut*.
- But since no two co-reachable states can accept the same words, we see that $\sum_{q_0 \in Q_0} \sum_{s \in \delta(q_0, p)} \zeta_{s,m} = 1$. We call the characteristic vector of a cut a *normaliser*.
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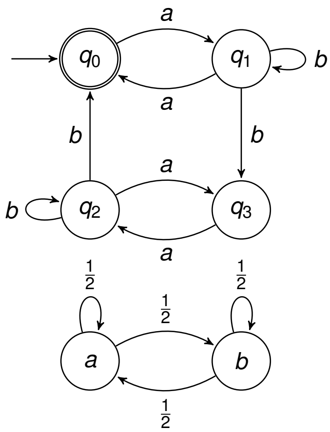

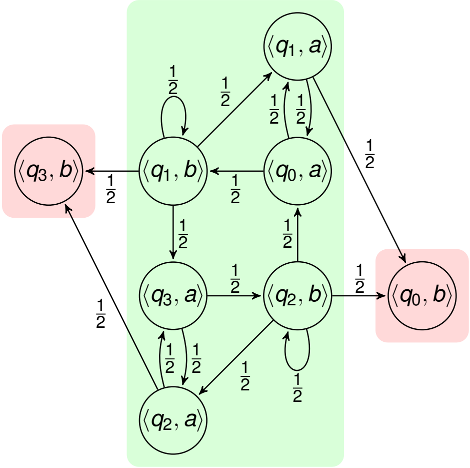

Calculating cuts

- Cuts are calculated using extenders
- Given a state q and a word w such that $\delta(q, w)$ is not a cut, an extender is a word v such that $\delta(q, v) \supseteq \{q, q'\}$, $q \neq q'$, and $\delta(q', w) \neq \emptyset$. Then $\delta(q, vw) \supsetneq \delta(q, w)$.

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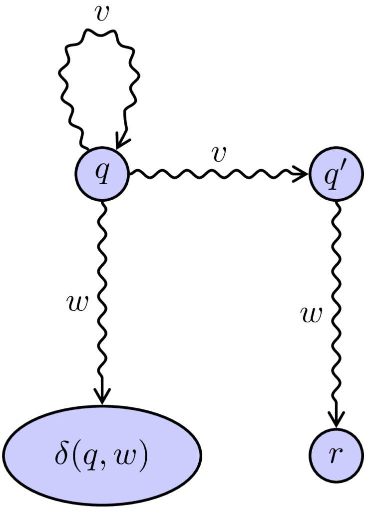
Extenders increase set size



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- By repeatedly calculating extenders we get a word p such that almost any infinite word prefixed by p is accepted

Pseudo-cuts

- Cuts remain cuts, cut vectors remain cut vectors
- Sums of cut vectors also invariant under transition matrices
- Matrix semigroups with constant spectral radius have invariant affine plane (Protasov, Voynov, 2017)

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- Matrix semigroups with constant spectral radius have invariant affine plane
- Find the affine plane with BFS on M^*
- $Co(d)$-pseudo-cuts are the vectors in the space orthogonal to the plane, intersected with $Co(d)$

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Further work

- Finitely ambiguous automata
- (Almost) complementation
- ...