On the Complexity of Reachability in Parametric MDPs

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Overview

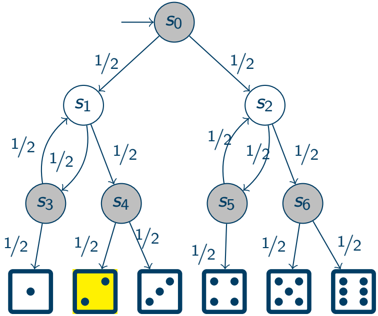
Parametric Markov models

Main contributions

Open problems

Conclusion

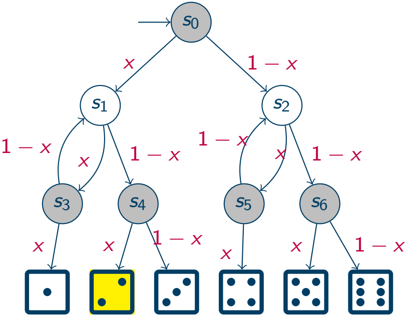
Knuth-Yao Die



Simulate 6-sided die by repeatedly throwing a **fair** coin

$Pr\left(reach\ ⚁\right) = 1/6$ ✓

Knuth-Yao Die with parametric coin

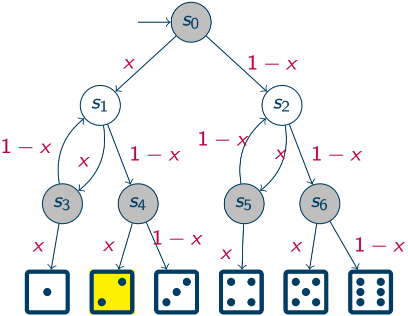


What if the coin is a little bit unfair?

$Pr\Big(reach$  $\Big) =$?

$x \in \left[\frac{49}{100}, \frac{51}{100}\right] \overset{?}{\Longrightarrow} Pr\Big(reach$  $\Big) \in \left[\frac{9}{60}, \frac{11}{60}\right]$

Knuth-Yao Die with parametric coin



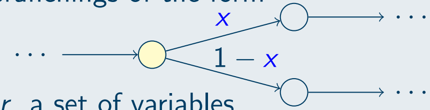
What if the coin is a little bit unfair?

$$Pr\left(reach\ \text{⚁}\right) = \frac{x^2-x^3}{x^2-x+1}$$

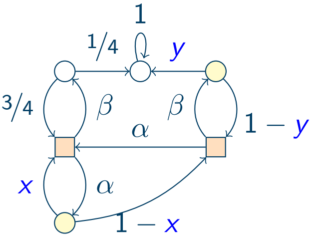
$$x \in \left[\frac{49}{100}, \frac{51}{100}\right] \overset{?}{\Longrightarrow} Pr\left(reach\ \text{⚁}\right) \in \left[\frac{9}{60}, \frac{11}{60}\right]$$

Definition (Daws '05, Lanotte et al. '07)

- A parametric MDP is an MDP that contains parametric probabilistic branchings of the form



where $x \in Var$, a set of variables.



Scheduler:
$\sigma : S \to Act$

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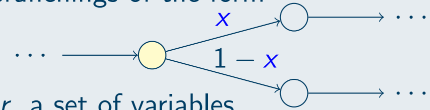
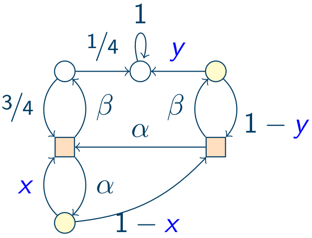


  where $x \in Var$, a set of variables.
- A parametric Markov chain is the special case without non-determinism.



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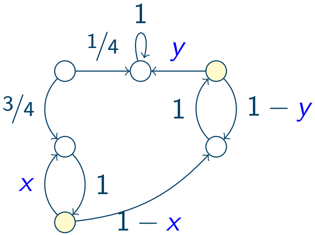
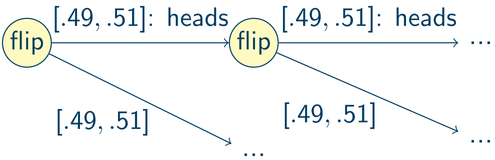


Why parametric models matter

- Exact probabilities often not available
- Interval models too pessimistic



- Extensive tool support
  - dedicated tools: *PARAM* [Hahn et al. '10], *PROPhESY* [Dehnert et al. '15]
  - general purpose prob. model checkers: *PRISM*, *STORM*, *ePMC*

Many open complexity questions

2 basic formal decision problems

- $\exists$Reach $\overset{def}{\iff}$ $\exists\vec{x}$: $Pr(reach ☺) \geq 1/2$? (for MCs)
- $\exists\forall$Reach $\overset{def}{\iff}$ $\exists\vec{x}\ \forall\sigma$: $Pr(reach ☺) \geq 1/2$? (for MDPs)

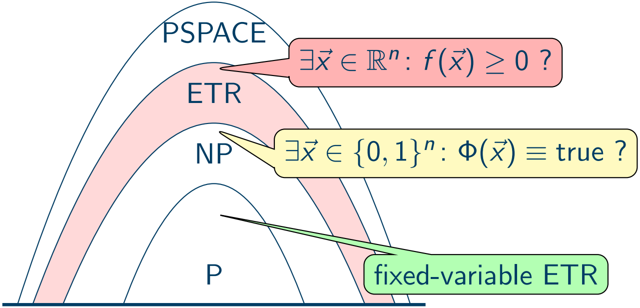
Theorem

	# params fixed	# params arbitrary	
$\exists$Reach	*in P [HBK'17]*	*ETR-complete*	$\leftarrow$ *Only* $\geq, \leq$
$\exists\forall$Reach	*in NP*	*ETR-complete*	$\leftarrow \geq, \leq, >, <$

- Further variants in paper

ETR as a complexity class

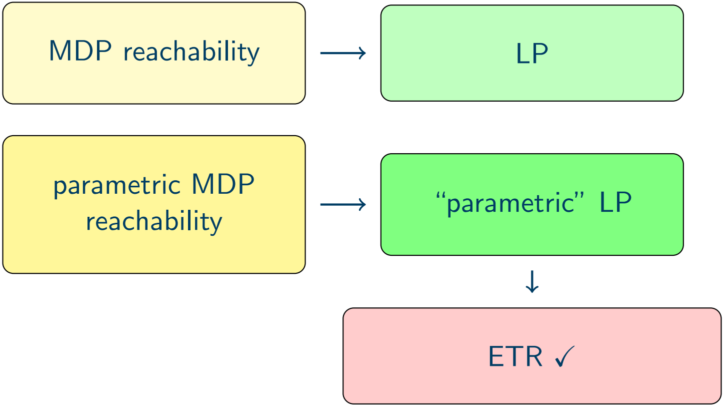
ETR = $\exists$-fragment of the FO theory $(\mathbb{R}, +, \cdot, 0, 1, <)$



Also ETR-complete

Several problems about Nash equilibria in 3-player games, planar graph drawing, and others regarding topology and geometry

∃∀Reach is in ETR



∃Reach is ETR-hard

ETR $\xrightarrow{\text{red.}}$ $\exists \vec{x} \in \mathbb{R}^n \colon f(\vec{x}) \geq 0?$ $\xrightarrow{\text{red.}}$ ∃Reach

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ETR $\xrightarrow{\text{red.}}$ $\exists \vec{x} \in [0,1]^n \colon f(\vec{x}) \geq 0?$ $\xrightarrow{\text{red.}}$ ∃Reach

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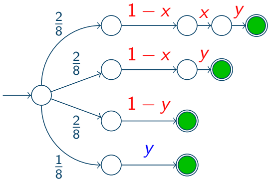
$$-2x^2y + y - 5 \geq 0$$



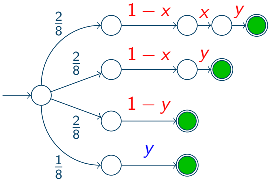
$\exists$Reach is ETR-hard

ETR $\xrightarrow{\text{red.}}$ $\exists \vec{x} \in [0,1]^n \colon f(\vec{x}) \geq 0?$ $\xrightarrow{\text{red.}}$ $\exists$Reach

$$-2x^2y + y - 5 \geq 0$$

(rewrite) $\Updownarrow$

$$2((1-x)xy + (1-x)y + (1-y) - 1) + y \geq 5$$



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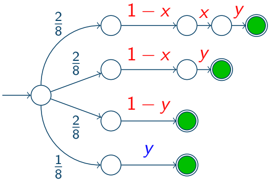
$$-2x^2y + y - 5 \geq 0$$

$$\text{(rewrite)} \ \Updownarrow$$

$$2((1-x)xy + (1-x)y + (1-y) - 1) + y \geq 5$$

$$\text{(scale)} \ \Updownarrow$$

$$\underbrace{\frac{2}{8}(1-x)xy + \frac{2}{8}(1-x)y + \frac{2}{8}(1-y) + \frac{1}{8}y}_{\text{sum of coefficients } \leq 1} \geq \frac{2 \cdot 1 + 5}{8}$$



This “trick” was first observed in [Chonev arXiv ’17]

Practice: often just a few parameters

Recall: fixed-variable ETR in P

	# params fixed	*# params arbitrary*
∃Reach	in P [HBK'17]	ETR-complete
∃∀Reach	**in NP**	ETR-complete

Lower complexity for fixed number of parameters ✓

∃∀Reach is in NP (fixed # of params)

- Use good parameters as polynomial certificate?
- Use a scheduler instead – which one?

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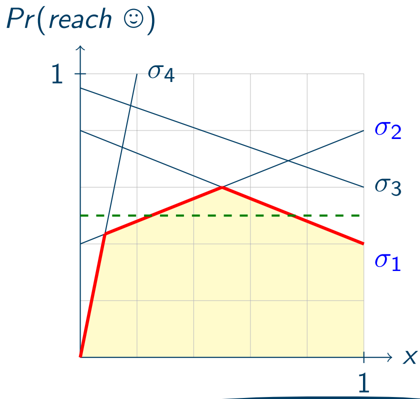

∃∀Reach is in NP (fixed # of params)

- Use good parameters as polynomial certificate?
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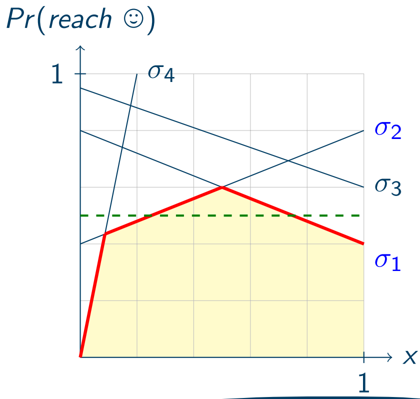
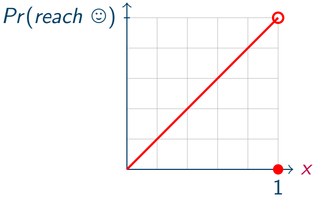
Check σ via fixed-param ETR query

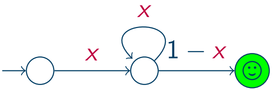
$$\bigwedge_{s,a} Pr^{\sigma}(\text{reach} ☺ \text{ from s}) \leq \sum_{s'} P(s,a,s') \underbrace{Pr^{\sigma}(\text{reach} ☺ \text{ from } s')}_{\text{fixed-param rational funct.}}$$



More refined results in paper

		# *params fixed*	# *params arbitrary*: well-defined, $[0,1]$	# *params arbitrary*: graph-preserving, $(0,1)$
pMC	$\exists\text{Reach}^{\geq/\leq}$	in P	— ETR-complete —	
	$\exists\text{Reach}^{>}$	"	NP-hard	$\exists\text{Reach}^{>}_{\text{wd}}$-complete
	$\exists\text{Reach}^{<}$	"	"	$\exists\text{Reach}^{>}_{\text{wd}}$-complete
pMDP	$\exists\exists\text{Reach}^{\geq/\leq}$	in NP	— ETR-complete —	
	$\exists\exists\text{Reach}^{>}$	"	— $\exists\text{Reach}^{>}_{\text{wd}}$-complete —	
	$\exists\exists\text{Reach}^{<}$	"	$\exists\text{Reach}^{<}_{\text{wd}}$-complete	$\exists\text{Reach}^{>}_{\text{wd}}$-hard
	$\exists\forall\text{Reach}^{\bowtie}$	in NP	— ETR-complete —	





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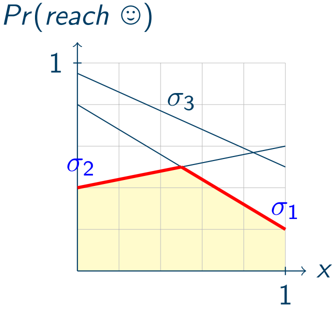
Additionally: Robust strategies, i.e. $\exists\sigma\forall\vec{x}$: $Pr(reach☺) \geq \frac{1}{2}$ under *deterministic memoryless* schedulers

1. Better complexity bounds

	# *params fixed*	...
∃Reach	in P [HBK'17]	...
∃∀Reach	in NP ← tight?	...

Can we show a *coNP* upper bound on fixed-param-∃∀Reach ?

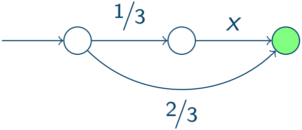


$\{\sigma_1, \sigma_2\}$ = minimal optimal scheduler set

∃ polynomially sized optimal scheduler set $\implies$ ∃∀Reach ∈ *coNP*

2. Connection pMC $\longleftrightarrow$ polynomials

- $Pr(reach\ ☺)$ is a polynomial for acyclic pMCs
- For which polynomials f is there a pMC with $Pr(reach\ ☺) = f$?



$\Rightarrow \frac{1}{3}x + \frac{2}{3}$

No pMC for $-2x^2y + y - 5$

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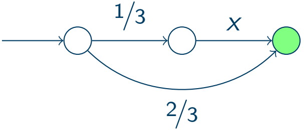

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For univariate f

If $f(x) \in (0,1)$ for $x \in (0,1)$, then there is a pMC with $Pr(\textit{reach} ☺) = f$.

Questions:

- How big is the resulting pMC? (lower bounds)
- What about multivariate polynomials?

Acyclic Markov chains with parametric x/1-x transitions are already hard, even for *graph-preserving* parameter valuations.

Any Boolean combination of *polynomial* constraints can be encoded into a pMC reachability problem.

A *fixed number of parameters* implies lower complexity for both pMCs & pMDPs.

Thank you for your attention!