Cellular Automata for the Self-stabilisation of Colourings and Tilings

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Reachability Problems – Brussels – Sept. 2019

Context & leading question

Turing, `The chemical basis of morophogenesis`, 1952.

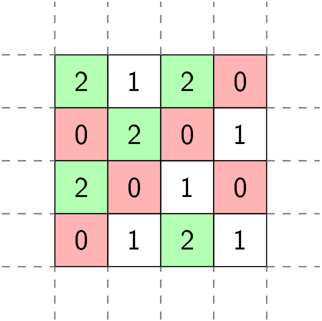
living organisms are **robust**
noise creates **patterns**

context:

discrete-time systems &
two-dimensional infinite grids

Λ : set of admissible configurations
self-stabilisation as a **reachability problem**:
start from Λ, perturb, return to Λ with **local rules** only?

Our problem in the context of cellular automata

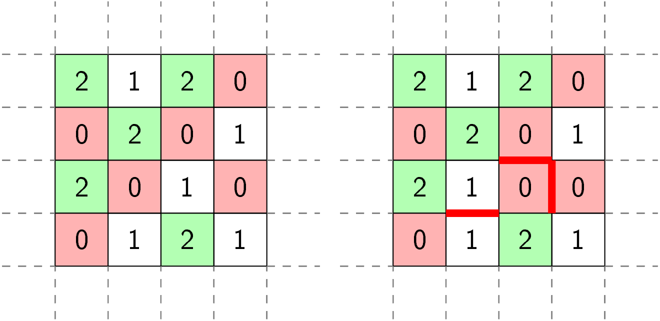


infinite grid of cells $\mathbb{Z}^2$
set of states $Q = \{1, \ldots, k\}$
set of configurations $E = Q^{\mathbb{Z}^2}$
neighbourhood $\mathcal{N} = \{n_1, \ldots, n_\nu\}$,
where $n_i \in \mathbb{Z}^2$
local function $f : Q^\nu \mapsto Q$
global function $F : E \mapsto E$

k-coulouring subshift

$$\Lambda_k = \{x \in E : c, c' \in \mathbb{Z}^2, ||c - c'||_1 = 1 \implies x_c \neq x_{c'}\}.$$

self-stabilisation & self-correction



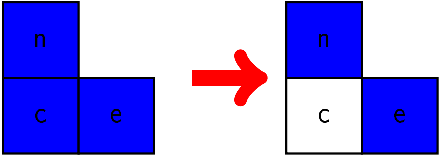
Λ : set of admissible configurations (here k-coulourings)

A CA F is **self-stabilising** on Λ if:

- elements of Λ are fixed points of F;
- starting from *any* configuration which is a *finite perturbation* of an element of Λ, we go back to Λ.

challenge: use only *local* information

the binary case $k = 2$



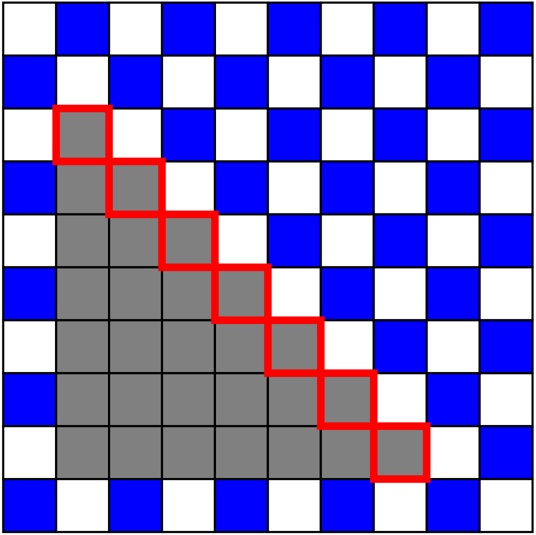
$$f(n, c, e) = \begin{cases} 1 - c & \text{if } n = c = e, \\ c & \text{otherwise.} \end{cases}$$

$$\forall (i, j) \in \mathbb{Z}^2, F(x)_{i,j} = f(x_{i,j+1}, x_{i,j}, x_{i+1,j}).$$

Theorem

F is self-correcting on Λ_2

proof of the binary case



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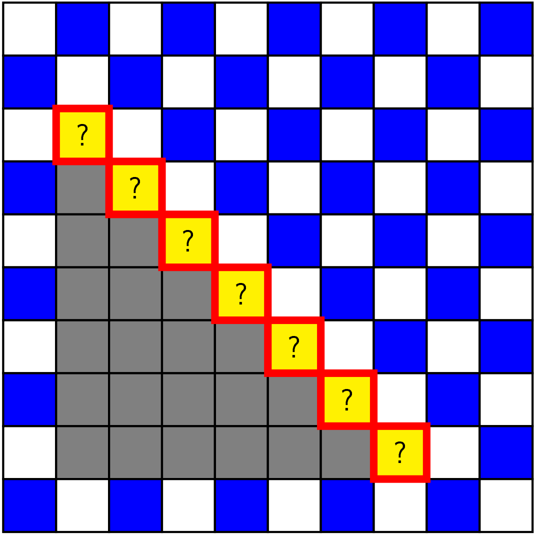

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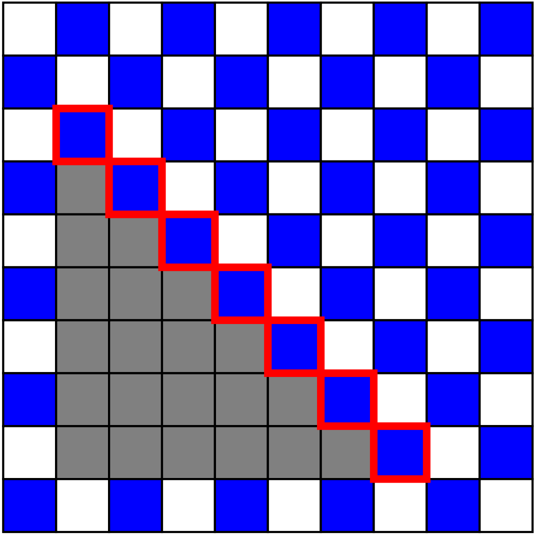
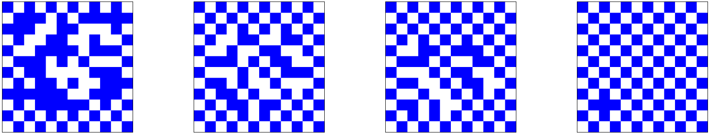


$k = 2$: stochastic and isotropic

$$\varphi(c, n, e, s, w) = \begin{cases} \text{minority}\,(c, n, e, s, w) & \text{with prob. } \alpha \\ c & \text{with prob. } 1 - \alpha \end{cases}$$



$t = 0$ $t = 2$ $t = 4$ $t = 10$

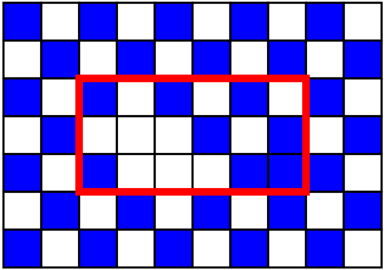
Theorem

The cellular automaton defined above is self-stabilising on Λ_2.

$k = 2$: stochastic and isotropic

Theorem

The cellular automaton defined above is self-stabilising on Λ_2.

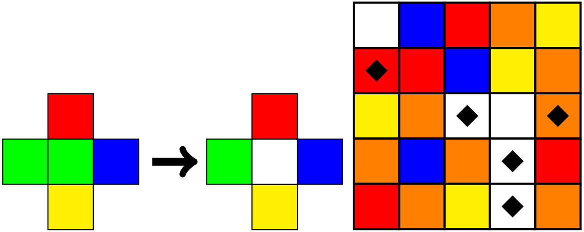


key:
inside the rectangle, we have an absorbing Markov chain

α-asynchronism ensures that there are no deadlocks...

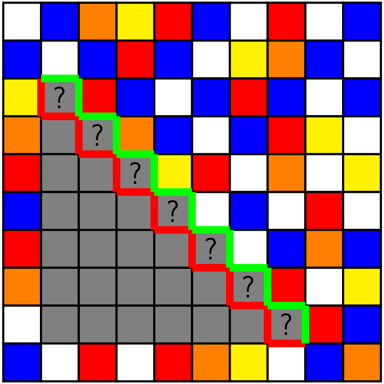
Five colors or more: $k \geq 5$

rule:



correction : take a colour that is not present among the colours of the four neighbours
local rule : apply the correction only for the cells which have a NE-error.

Five colors or more

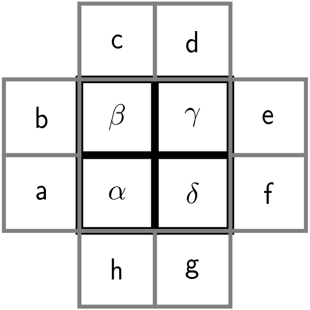


proof :
some cells will change their states, others will remain

Once again, we enclose the erros in a triangle and show that it can only decrease.

The four-colour case: $k = 4$

Local corrections by synchronising squares of cells
prop:
for all the settings $a, \ldots, h$ there exists values for $\alpha, \beta, \gamma, \delta$ s.t. the pattern has no conflicts



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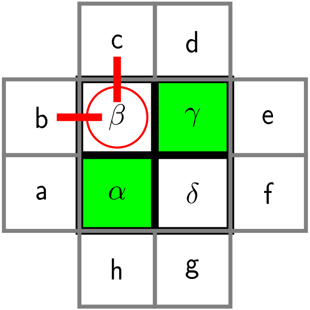
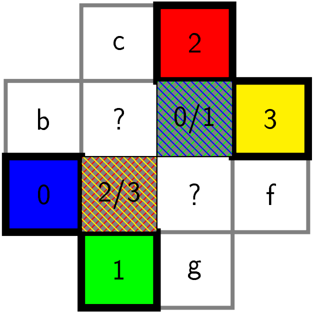


- if we can set $\alpha = \gamma$

β and δ have at most three surrounding colours ; choose the remaining colour.

the four-coulour case (continued)

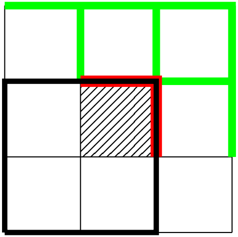


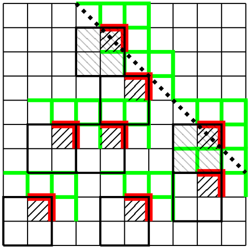
cells cannot have a four-coloured neighb.

a conflict is not avoidable only if a cell has necesssarily *four* neighbouring colours

if we try to create such a situation,
we have $b \neq c$ and $g \neq f$,
we find a solution in each case

applying the corrections





Apply the correction to the four cells *simultaneously* if:

- red borders contain one or two mistakes,
- green borders have no mistake.

the shape of the "neighbourhood" ensures that no conflicting updates occur

corrections by diagonals (sweeping)

The isotropic and stochastic case: $k = 4$

von Neumann neighb. (nearest neighb.), rule φ:

- if the four colours are not present (in NESW), choose one missing colour
- otherwise, choose one colour randomly (different from yourself)

local rule f : apply ϕ with probability $\alpha = 1/2$ and identity with probability $1 - \alpha = 1/2$

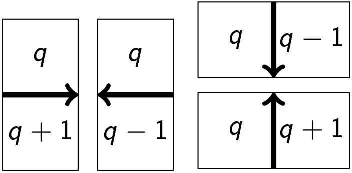
conjecture:
f is self-correcting on Λ_4

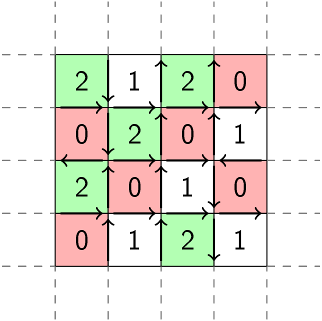
simulations show **amazing** convergence properties... yet to be fully understood

A challenge : three-colour case $k = 3$

conjecture : extra states are needed to define a self-correcting rule

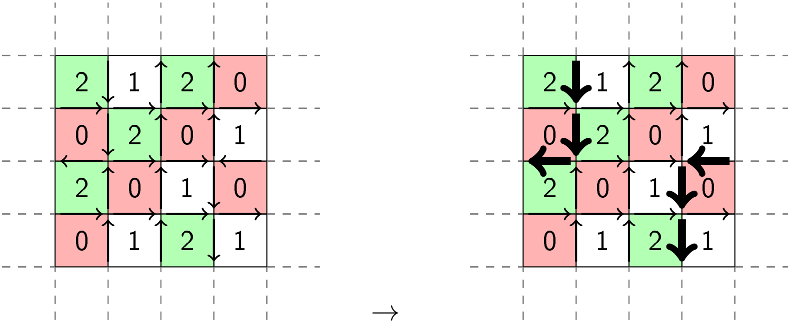
key idea by Irène M. : conversion to the six-vertex model





error-free 3-colouring $\equiv$ two incoming & outgoing arrows per node

elementary properties of the six-vertex model

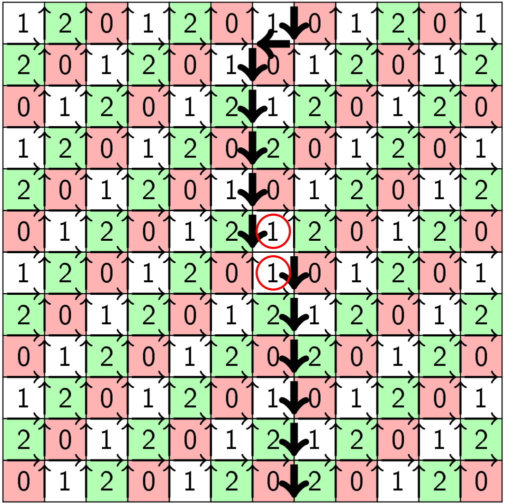


error-free 3-colouring $\equiv$ two incoming & outgoing arrows per node

knowledge of South and West arrows is sufficient $\rightarrow$ distinguished

they necessarily form a sequence of connected paths

no "limited-horizon correction"



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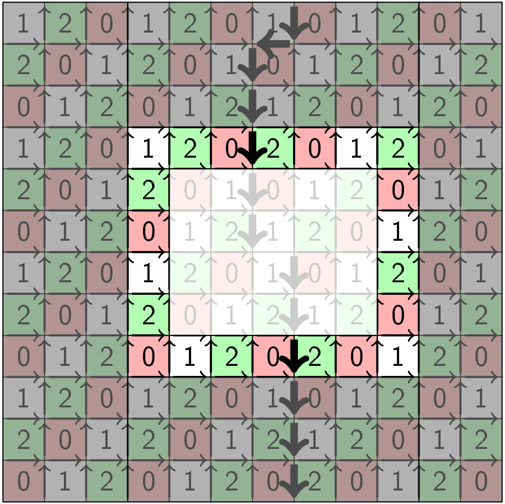

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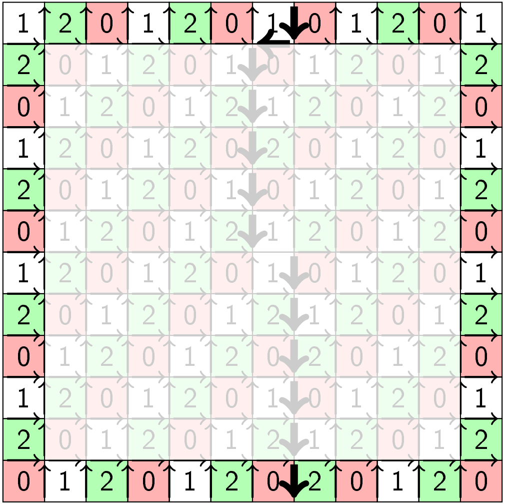

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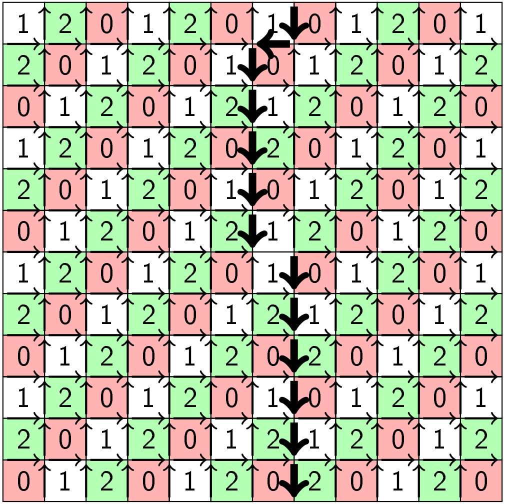

difficult to imagine how to do this without extra symbols...

perspectives

- first exploration of self-stabilisation properties in cellular automata...
- explore the case $k = 3$ with extra states ; understand more precisely the absence "limited-horizon" correction
- **deterministic vs. stochastic** cellular automata: simplicity, isotropy, computing abilities...

- generalise from k-colourings to subshifts
- generalise to more general tilings or graphs...