Coverability is Undecidable in One-dimensional Pushdown Vector Addition Systems with Resets

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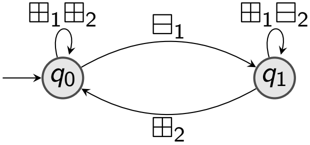
Reachability Problems 2019

Two-dimensional VASS



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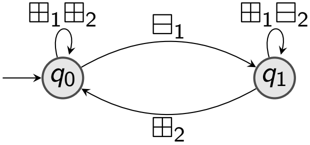

Configurations: $Q \times \mathbb{N} \times \mathbb{N}$

$\boxplus_i$ increment counter i

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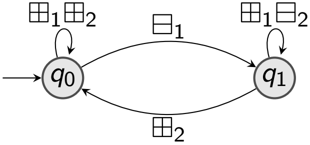

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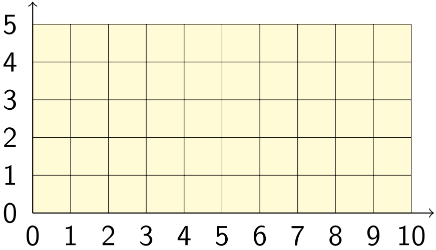

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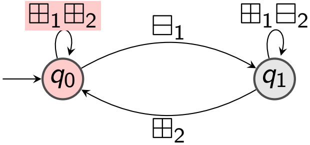

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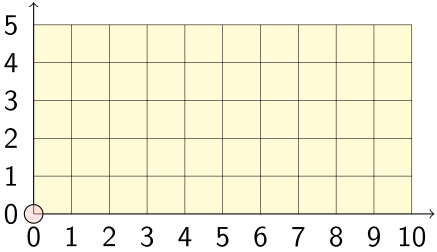

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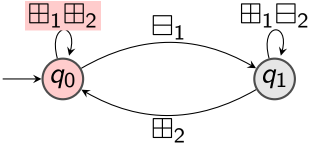

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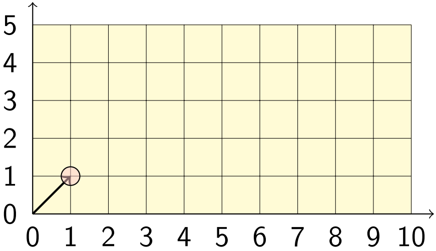

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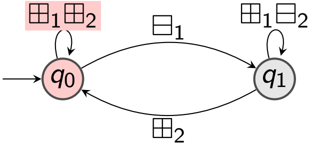

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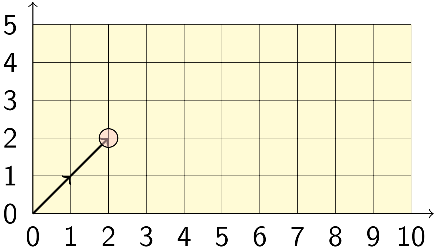

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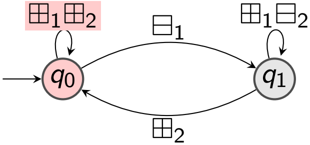

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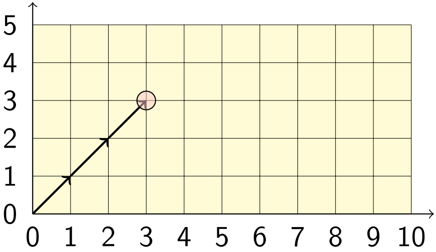

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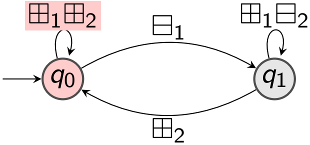

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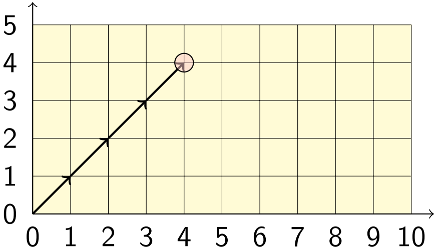

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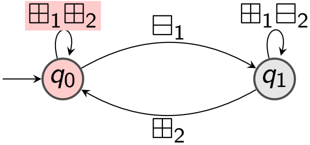

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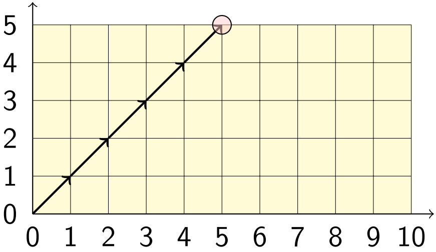

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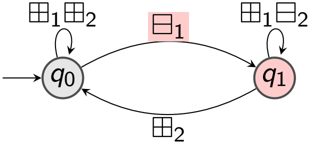

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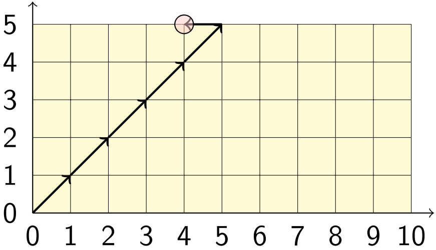

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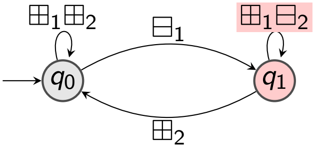

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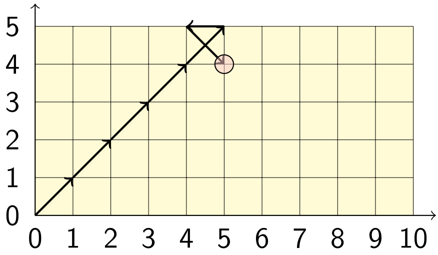

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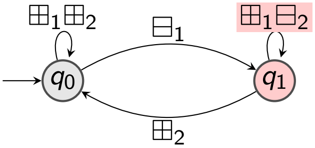

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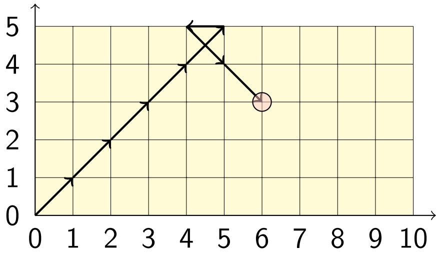

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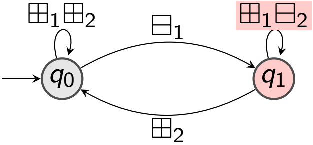

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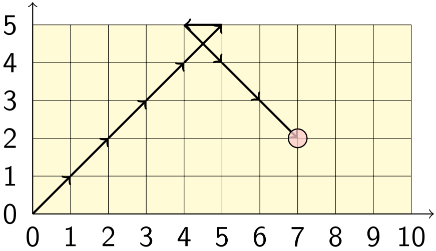

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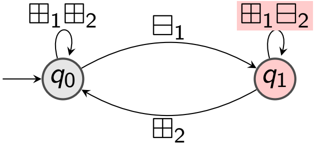

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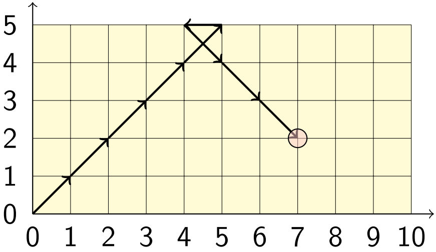

Reachability problem

Given Configuration (q, m, n)

Question $(q_0, 0, 0) \xrightarrow{*} (q, m, n)$?

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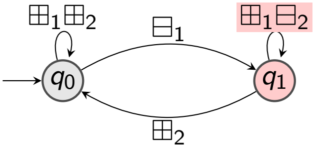

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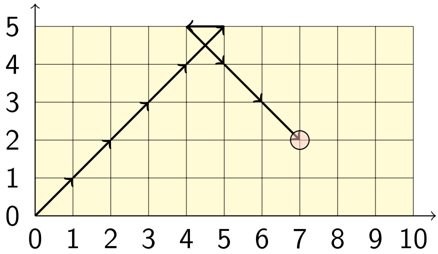


Reachability problem

Given Configuration (q, m, n)

Question $(q_0, 0, 0) \xrightarrow{*} (q, m, n)$?

Coverability problem

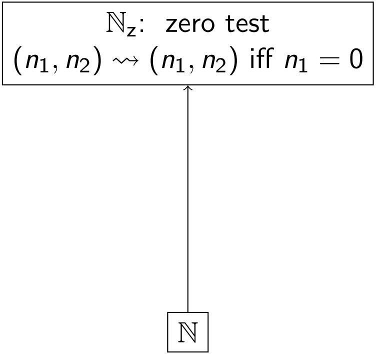
Given State q

Question $(q_0, 0, 0) \xrightarrow{*} (q, m, n)$ for some $m, n \in \mathbb{N}$?

Extensions of $\mathbb{N}$ counters

$\boxed{\mathbb{N}}$

Extensions of $\mathbb{N}$ counters



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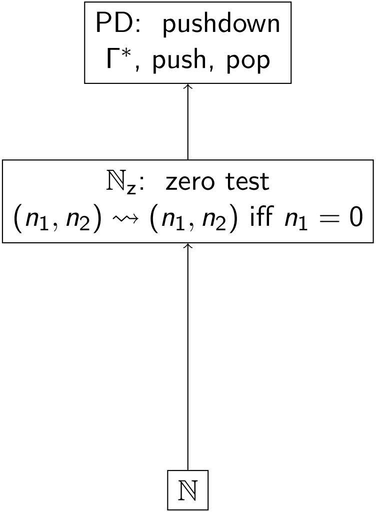

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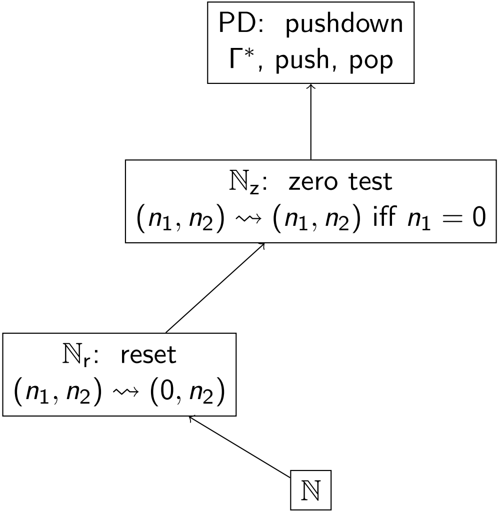

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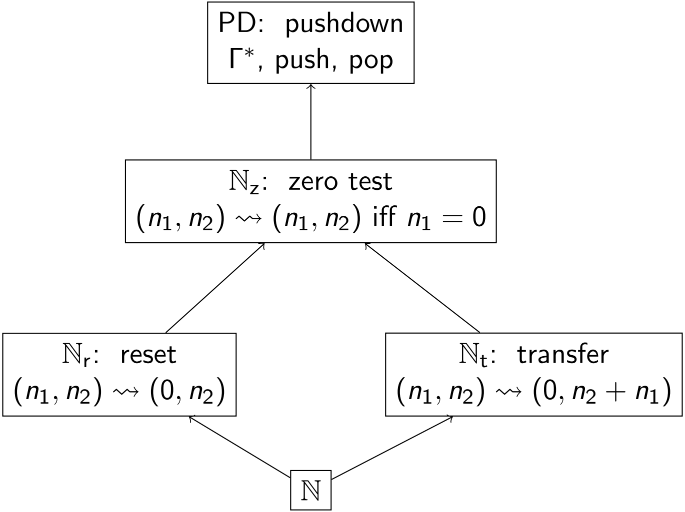

Decidability of coverability and reachability

$\mathbb{N}$	$\mathbb{N}_r$	$\mathbb{N}_t$	$\mathbb{N}_z$	PD	Cov./Reach.
					$\mathbb{N}$
					$\mathbb{N}_r$
					$\mathbb{N}_t$
					$\mathbb{N}_z$
					PD

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$\mathbb{N}$	$\mathbb{N}_r$	$\mathbb{N}_t$	$\mathbb{N}_z$	PD	Cov./Reach.
					$\mathbb{N}$
					$\mathbb{N}_r$
			U/U		$\mathbb{N}_t$
					$\mathbb{N}_z$
					PD

Finkel and Sutre, STACS 2000

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					$\mathbb{N}$
					$\mathbb{N}_r$
			U/U	U/U	$\mathbb{N}_t$
			U/U	U/U	$\mathbb{N}_z$
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$\mathbb{N}$	$\mathbb{N}_r$	$\mathbb{N}_t$	$\mathbb{N}_z$	PD	Cov./Reach.
					$\mathbb{N}$
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	D/D	D/D	D/D		$\mathbb{N}_r$
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		D/D	U/U	U/U	$\mathbb{N}_t$
			U/U	U/U	$\mathbb{N}_z$
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D/D	D/D	D/D	D/D	D/??	$\mathbb{N}$
	D/D	D/D	D/D		$\mathbb{N}_r$
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Leroux, Sutre, and Totzke, ICALP 2015

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		D/D	U/U	U/U	$\mathbb{N}_t$
			U/U	U/U	$\mathbb{N}_z$
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Theorem

Coverability is undecidable for $\text{PD} + \mathbb{N}_r$.

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- Still decidable with $\mathbb{N}_z$ instead of PD

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Theorem

Coverability is undecidable for $\mathrm{PD} + \mathbb{N}_r$.

- Still decidable with $\mathbb{N}_z$ instead of PD
- Undecidability that must exploit pushdown

Ingredient I

Theorem (Minsky 1961)

Reachability is undecidable in $\mathbb{N}_{\mathsf{z}} + \mathbb{N}_{\mathsf{z}}$.

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$$(n_1, n_2) \mapsto 2^{n_1} \cdot 3^{n_2}$$

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Operation in $\mathbb{N}_z + \mathbb{N}_z$	On encoding
increment n_1	multiply by 2

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Operation in $\mathbb{N}_z + \mathbb{N}_z$	On encoding
increment n_1	multiply by 2
decrement n_1	divide by 2
zero test n_1	verify $n_1 \not\equiv 0 \bmod 2$

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Operation in $\mathbb{N}_z+\mathbb{N}_z$	On encoding
increment n_1	multiply by 2
decrement n_1	divide by 2
zero test n_1	verify $n_1 \not\equiv 0 \bmod 2$
increment n_2	multiply by 3

Ingredient I

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Encoding

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Operation in $\mathbb{N}_z + \mathbb{N}_z$	On encoding
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decrement n_1	divide by 2
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Initial value $2^0 \cdot 3^0 = 1$
Final value $2^0 \cdot 3^0 = 1$

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$$(n_1, n_2) \mapsto 2^{n_1} \cdot 3^{n_2}$$

Operation in $\mathbb{N}_z+\mathbb{N}_z$	On encoding	Symbol
increment n_1	multiply by 2	m_2
decrement n_1	divide by 2	d_2
zero test n_1	verify $n_1 \not\equiv 0 \bmod 2$	t_2
increment n_2	multiply by 3	m_3
decrement n_2	divide by 3	d_3
zero test n_2	verify $n_2 \not\equiv 0 \bmod 3$	t_3

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$\Delta = \{m_2, d_2, t_2, m_3, d_3, t_3\}$

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Relations

$$R_{m_f} = \{(n,\ f \cdot n) \mid n \in \mathbb{N}\}$$
$$R_{d_f} = \{(f \cdot n,\ n) \mid n \in \mathbb{N}\}$$
$$R_{t_f} = \{(n, n) \mid n \in \mathbb{N}, n \not\equiv 0 \bmod f\}$$

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Reformulation

Given Regular language $M \subseteq \Delta^*$

Question Is there a word $x_1 \cdots x_\ell \in M$ with $(1, 1) \in R_{x_1} R_{x_2} \cdots R_{x_\ell}$?

$\Delta = \{m_2, d_2, t_2, m_3, d_3, t_3\}$ $RS = \{(m,n) \mid \exists k\colon (m,k) \in R,\ (k,n) \in S\}$

Relations

$R_{m_f} = \{(n,\ f \cdot n) \mid n \in \mathbb{N}\}$

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Problem

Implementing R_{m_f}, R_{d_f}, R_{t_f} directly likely impossible!

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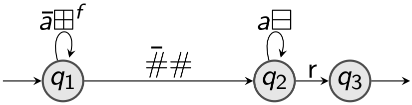

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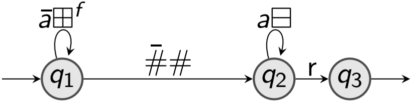

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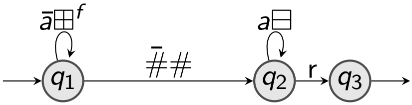

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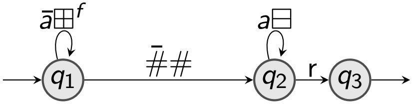

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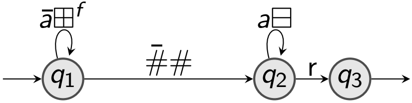

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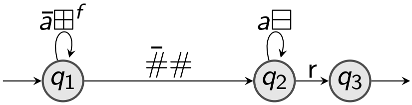

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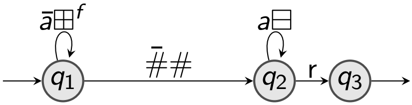

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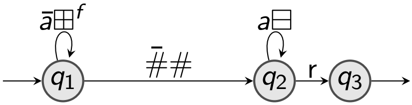

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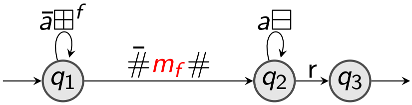

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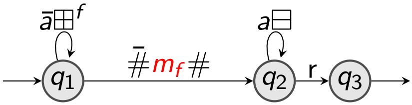

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\begin{array}{ccccccccc}
x_0 & \color{red}{R_1} & x_1 & \color{red}{R_2} & x_2 & & & & \\
 & & & & \color{red}{\vee} & & & & \\
\| & & & & x'_2 & \overrightarrow{R_3} & \cdots & \overrightarrow{R_n} & x_n \\
 & & & & & & & & \| \\
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 & & & & \vee & & & & \\
\| & & & & & & & & \\
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 & & & & \color{red}{\rotatebox{90}{$\geq$}} & & & & \| \\
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- Complexity of cov. and reach. for $\mathbb{N}_z + \mathbb{N}_r$, $\mathbb{N}_t + \mathbb{N}_t$, $\mathsf{PD} + \mathbb{N}$?

Conclusion

Theorem

Coverability in $\mathsf{PD} + \mathbb{N}_r$ *is undecidable.*

- settles last case of coverability in extensions of 2-VASS
- undecidability that really exploits pushdown

Fun fact

There exists a language K so that:

- intersection with K is decidable for one-counter languages, but
- intersection with K is undecidable for context-free languages.

Open problems

- Is reachability decidable in $\mathsf{PD} + \mathbb{N}$?
- Complexity of cov. and reach. for $\mathbb{N}_z + \mathbb{N}_r$, $\mathbb{N}_t + \mathbb{N}_t$, $\mathsf{PD} + \mathbb{N}$?
  ($\mathbb{N} + \mathbb{N}$: PSPACE for binary encoding (Blondin et. al. 2015),
  NL for unary encoding (Englert et. al. 2016))