m-Eternal Domination Number of Cactus Graphs

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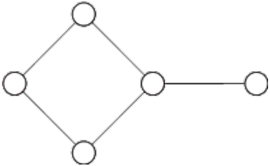
- **Question**: Given a graph G, what is the minimum number of guards required to defend it against any sequence of attacks?

m-Eternal Domination

Input graph G:



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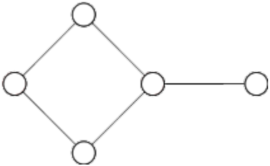
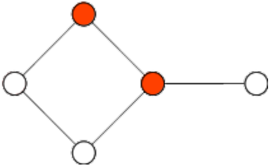
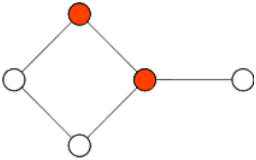


Suppose we choose this initial configuration of guards (red):



Example



Turn 1

- In each turn, the attacker chooses one vertex to attack.
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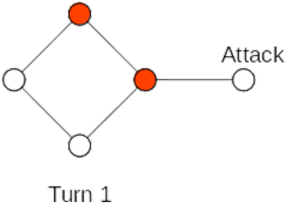

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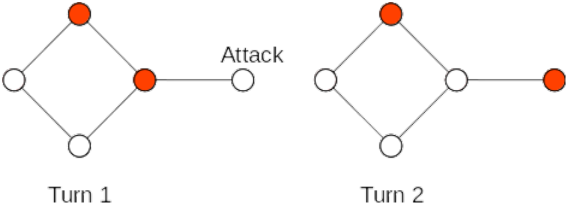

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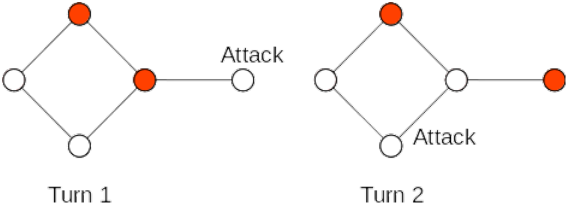

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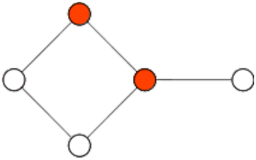

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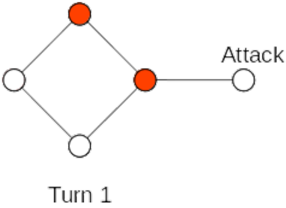

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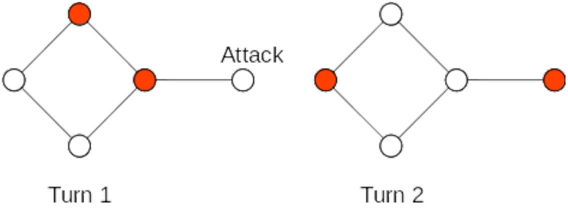

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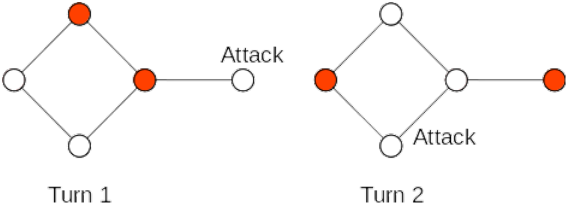

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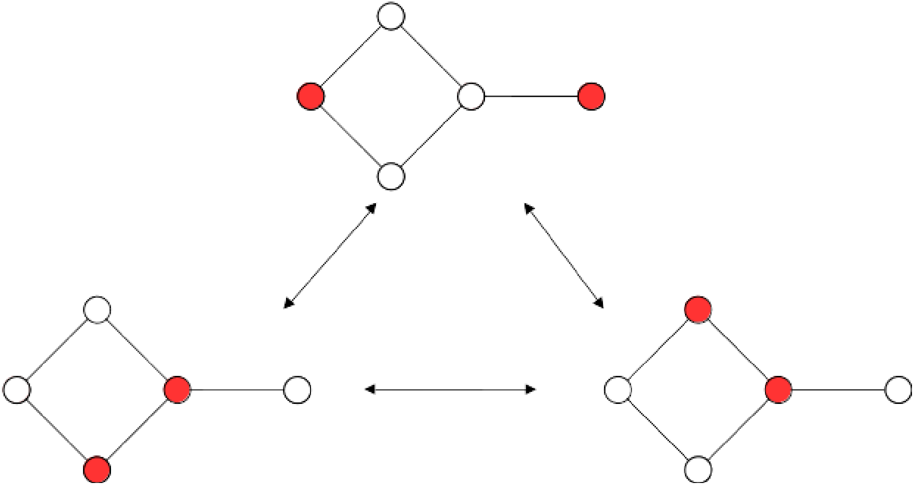
Example

Three configurations of two guards are sufficient to defend against any sequence of attacks.



Known results

m-Eternal Domination - decision variant:

Input: Graph G, integer k
Output: Can k guards defend G against any possible sequence of attacks?

- Known to be NP-Hard
- Lies in EXPTIME
- Unknown whether lies in PSPACE

Known results

- Solved for cycles, complete graphs, complete bipartite graphs [Goddard et al. 2005]
- Linear algorithm for trees [Klostermeyer, MacGillivray, 2009]
- Linear algorithm for interval graphs [Rinemberg, Soulignac, 2019]

Let $\gamma_m^\infty(G)$ be the minimum number of guards required to defend G against any sequence of attacks.

Theorem (Henning, Klostermeyer and MacGillivray, 2017)

Let G be a connected graph with minimum degree $\delta(G) \geq 2$ which has $n \neq 4$ vertices. Then $\gamma_m^\infty(G) \leq \lfloor (n-1)/2 \rfloor$ and this bound is tight.

Known results

Theorem (Finbow, Messinger and van Bommel, 2015)

For $n \geq 2$,

$$\gamma_m^\infty(P_3 \square P_n) \leq \lceil 6n/7 \rceil + \begin{cases} 1 & \text{if } n \equiv 7, 8, 14 \text{ or } 15 \pmod{21} \\ 0 & \text{otherwise} \end{cases}$$

Theorem (van Bommel, van Bommel, 2016)

$$\lfloor \frac{6n+9}{5} \rfloor \leq \gamma_m^\infty(P_5 \square P_n) \leq \lfloor \frac{4n+4}{3} \rfloor$$

- More known results can be found in *Protecting a Graph with Mobile Guards* [Klostermeyer, Mynhardt, 2015]

Our results

Theorem

Let G be a Christmas cactus graph. Then there exists a linear-time algorithm which computes the minimum required number of guards to defend G.

Christmas cactus graph

Definition

Cactus graph is a connected graph in which every edge lies on at most one cycle.

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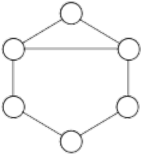
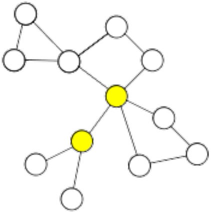
Christmas cactus graph is a cactus graph, in which removal of any vertex splits the graph into at most two connected components.



Not a cactus



Cactus
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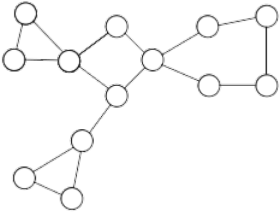


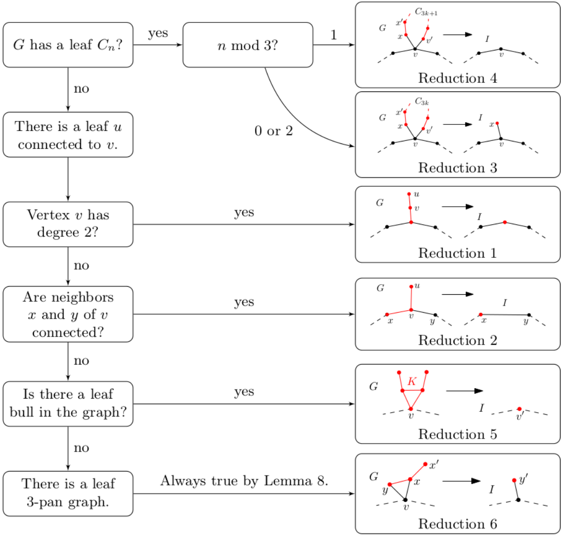
Christmas cactus

Algorithm

- Algorithm works as follows:
  - If G is a trivial case (cycle, K_2, K_1), output result.
  - Otherwise, perform one of possible reductions, which decrease γ_m^∞ by a constant known amount. The result of the reduction is always a smaller Christmas cactus graph.
  - Repeat with the reduced graph.

Overview of the reductions

If G is not trivial (cycle, K_2, K_1):



Idea of the proof

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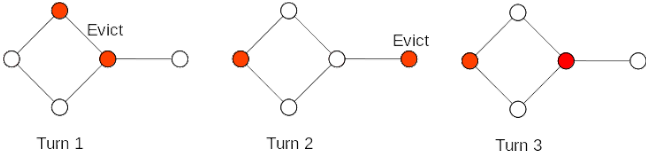
  - Instead of attacking, the attacker can pick an edge on a cycle or a vertex, which has to be evicted by guards.
  - 

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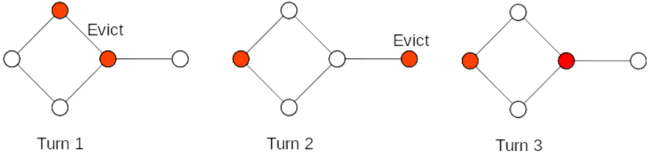
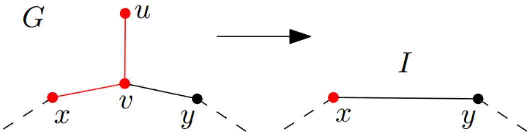
  - 

- **Lower bound:** allow multiple guards on one vertex

Idea of the proof

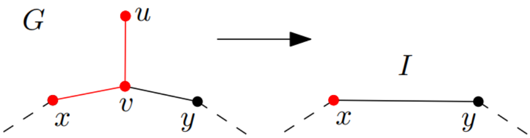
- The relationship for the optimal number of guards between these games is as follows:
- **m-eternal domination with multiple guards on one vertex**
  $\leq$
  m-eternal domination
  $\leq$
  m-eternal domination with eviction

Example of a reduction



- Upper bound
  - Assume an optimal strategy on I for m-eternal domination with eviction.
  - Extend it to G while adding one guard.
  - Show how to use the new guard to defend u and v.
  - Show how to evict edges $\{x, v\}$, $\{v, y\}$ and vertices u, v.
  - Show how to simulate a guard passing through $\{x, y\}$ in I, so the strategy is applicable in G.

Example of a reduction



- Lower bound
  - Assume an optimal strategy on G for m-eternal domination with multiple guards on one vertex.
  - Adapt the strategy of G for I.
  - Contract edges $\{u, v\}$ and $\{v, y\}$. Any guard moving along a contracted edge does not move.
  - Show that there is a guard which never leaves y and is not necessary and therefore can be removed.
  - The resulting strategy defends I and uses one less guard than in G.

Summary

- Linear algorithm for Christmas cactus graphs
- Upper bound on the optimal number of guards for cactus graphs based on a decomposition into Christmas cactus graphs.

Open problems

- More efficient (than EXPTIME) algorithm for graphs with treewidth 2.
- Algorithm parameterized by treewidth.
- Does the decision variant of the problem lie in PSPACE? Is it EXPTIME hard?
- Can we bound the number of required guard configurations in an optimal strategy?

The end

Thank you for your attention!