Termination of Affine Loops over the Integers

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RP

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Affine Loops

We study programs of the form

$$\mathtt{P} : \textit{while } (\boldsymbol{C}\boldsymbol{x} > \boldsymbol{d}) \textit{ do } \boldsymbol{x} := \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

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This loop has eigenvalues $e_1 = -1 - \sqrt{17}, e_2 = -1 + \sqrt{17}$ with eigenvectors $\boldsymbol{v}_1 = (-1 - \sqrt{17}, 4)$, $\boldsymbol{v}_2 = (-1 + \sqrt{17}, 4)$. Note that

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    p = r = b + (2 * PTHRESH);
    if (r >= t) p = r = t;      /* too short to care about */
    else {
        while (((cmp(aTHX_ *(p-1), *p) > 0) == sense) &&
               ((p -= 2) > q)) {}
        if (p <= q) {
            /* b through r is a (long) run.
            ** Extend it as far as possible. */
            p = q = r;
            while (((p += 2) < t) &&
                   ((cmp(aTHX_ *(p-1), *p) > 0) == sense)) q = p;
            r = p = q + 2;      /* no simple pairs, no after-run */
        }
    }
    if (q > b) {                /* run of greater than 2 at b */
        gptr *savep = p;
        p = q += 2;
        /* pick up singleton, if possible */
        if ((p == t) &&
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            savep = r = p = q = last;
        p2 = NEXT(p2) = p2 + (p - b); ++runs;
        if (sense)
            while (b < --p) {
                const gptr c = *b;
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    while (q < p) {             /* simple pairs */
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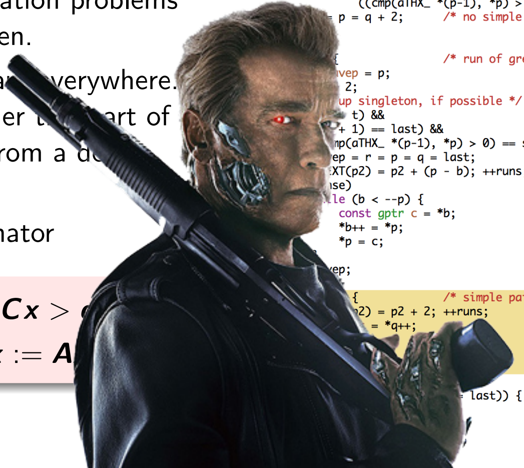
           ep = r = p = q = last;
           XT(p2) = p2 + (p - b); ++runs;
           nse)
           le (b < --p) {
             const gptr c = *b;
             *b++ = *p;
             *p = c;

           ep;

           {                /* simple pairs */
           2) = p2 + 2; ++runs;
           = *q++;

           last)) {
```



while $(\boldsymbol{C}\boldsymbol{x} > c$ …

do $\boldsymbol{x} := \boldsymbol{A}$ …

Affine Loops & Linear Recurrence Sequences

A Linear Recurrence Sequence (LRS) $\mathbf{u} = \langle u_n \rangle_{n=1}^{\infty}$ is a sequence defined by a recursion of the form

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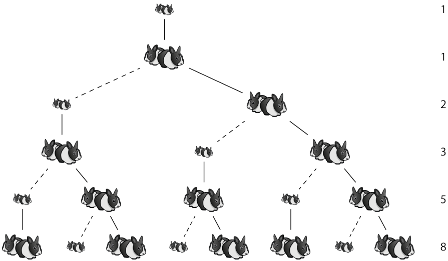

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For a loop

$$\texttt{P} : \textit{while } (\boldsymbol{c}\boldsymbol{x} > \boldsymbol{d}) \textit{ do } \boldsymbol{x} := \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

it can be readily seen that

$$u_n = \begin{pmatrix} \boldsymbol{c} & -\boldsymbol{d} \end{pmatrix} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix}^n \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}$$

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- **Khinchine:** There exists $W \in \mathbb{N}$ s.t. any convex set $C \subseteq \mathbb{R}^d$ of width at least W contains an integer point.

□

Finding Non-terminating Integer Points

Proof Sketch.

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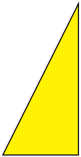

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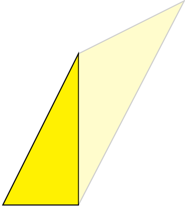

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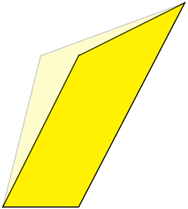

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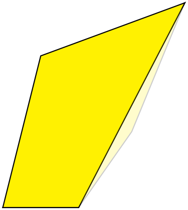

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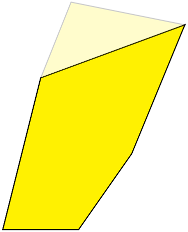

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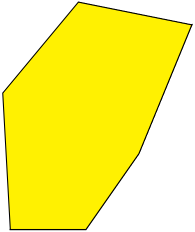

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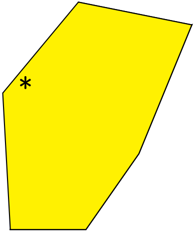

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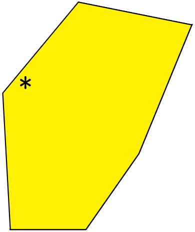

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$$\mathtt{P} : \textit{while } (\boldsymbol{Bx} > \boldsymbol{b}) \textit{ do } \boldsymbol{x} := \boldsymbol{Ax} + \boldsymbol{a}$$

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□

Ranking Functions: They don't always exist

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Solution.
This loop has **doesn't** have a linear ranking function. □

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Thank You!