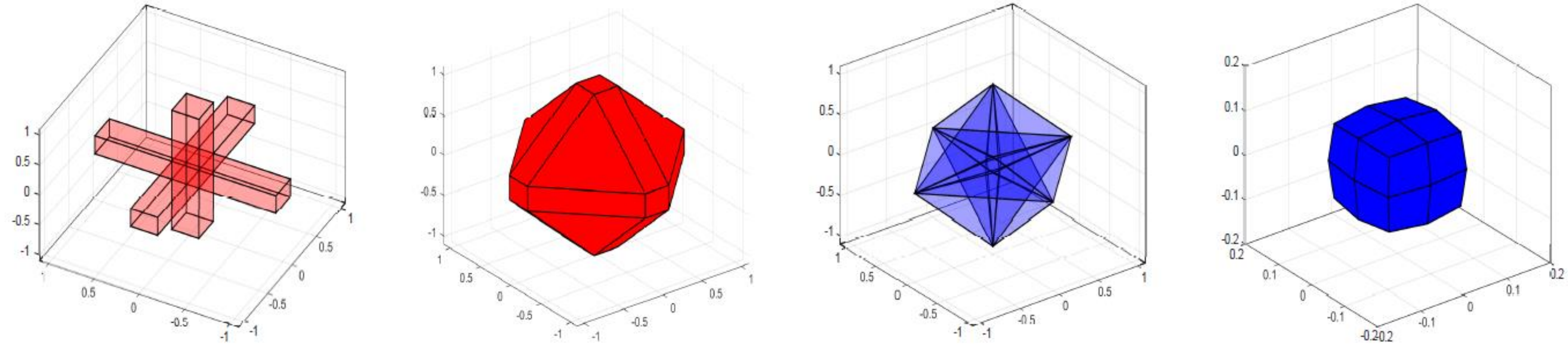
Path-Complete Reachability for Switching Systems



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RP'19, September 12

Outline

- Introduction and context
- Path Complete Graphs and induced Lyapunov Functions (LFs)
- Polyhedral path complete Lyapunov functions (+ motivating example)
- Partial liftings
- Conclusions

<u>Note</u>: The talk is about stability analysis (or positively invariant sets) for switching systems. The exposition is via Lyapunov theory (alternatively: backward/forward reachability maps inclusions).

Part of these results is going to be published in the proceedings of the IEEE Conference on Decision and Control, 2019

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- Introduction and context
- Path Complete Graphs and induced Lyapunov Functions (LFs)
- **Polyhedral path complete Lyapunov functions (+ motivating example)**
- **Partial liftings**
- Conclusions

Lyapunov Functions (LFs)

Dynamical Systems: $x(t+1) = f(x(t)), \quad t \geq 0, \quad x(0) \in \mathbb{R}^n$

Lyapunov conditions:

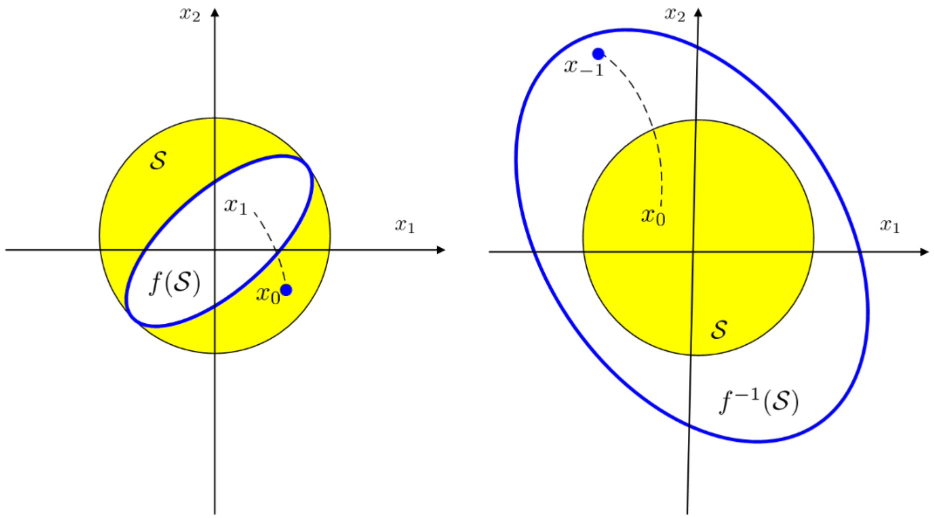
$$V(x) > 0 \quad x \neq 0$$
$$V(x) = 0 \quad x = 0$$
$$V(f(x)) \leq \varepsilon V(x) \quad x \in \mathcal{X}$$

(Lyapunov decrease conditions) $\varepsilon \in [0, 1]$

Sets interpretation:

$$\mathcal{S} = \{x : V(x) \leq 1\} \subseteq \mathcal{X}$$

$f(\mathcal{S}) \subseteq \mathcal{S}$ or $f^{-1}(\mathcal{S}) \supseteq \mathcal{S}$



Lyapunov Functions (LFs)

Dynamical Systems: $x(t+1) = f(x(t)), \quad t \geq 0, \quad x(0) \in \mathbb{R}^n$

Challenge: (i) Find function V, (ii) evaluate the decrease condition

Lyapunov conditions:

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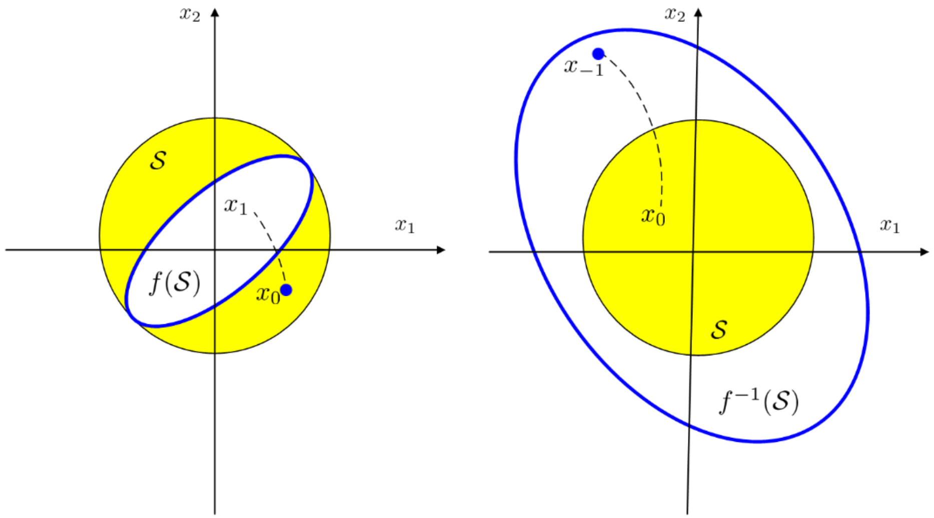


Methods to find LFs (invariant sets)

Parameterizations – different templates for function V(.)

quadratic, sum of squares, piecewise linear, max-of-quadratics, min-of-quadratics, homogeneous, ….

Multiple Lyapunov functions – parameterization of the decrease conditions

Example: $x(t+1) = A_i x(t), \;\; i \in \{1,2\}$

Conditions:

$$\begin{aligned} V_a(A_1 x) &\leq V_a(x) \\ V_b(A_1 x) &\leq V_a(x) \\ V_a(A_2 x) &\leq V_b(x) \\ V_b(A_2 x) &\leq V_b(x) \end{aligned} \implies V(x) = \min\{V_a(x), V_b(x)\}$$

Reachability maps and invariance:

Iterative backward / forward reachability

maps provide invariant sets (sublevel sets are LFs)

Methods to find LFs (invariant sets)

Parameterizations – different templates for function V(.)

quadratic, sum of squares, piecewise linear, max-of-quadratics, min-of-quadratics, homogeneous, ….

Challenge: to have complex enough templates and solvable decrease conditions

Multiple Lyapunov functions – parameterization of the decrease conditions

Example: $x(t+1) = A_i x(t), \;\; i \in \{1,2\}$

Conditions:

$$V_a(A_1 x) \leq V_a(x)$$
$$V_b(A_1 x) \leq V_a(x)$$
$$V_a(A_2 x) \leq V_b(x)$$
$$V_b(A_2 x) \leq V_b(x)$$

$$\Longrightarrow \quad V(x) = \min\{V_a(x), V_b(x)\}$$

Reachability maps and invariance:

Iterative backward / forward reachability

maps provide invariant sets (sublevel sets are LFs)

Challenge: complexity grows with number of iterations

Path Complete Lyapunov Functions

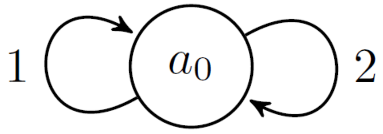
Constrained Switching Systems:

Let $\mathcal{A} = \{A_1, ..., A_N\}$, a labeled directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$

$$x(t+1) = A_{\sigma(t)} x(t)$$
$$z(t+1) \in \mathrm{Out}(z(t), \mathcal{G}(\mathcal{V}, \mathcal{E}))$$
$$(z(t), z(t+1), \sigma(t)) \in \mathcal{E}$$

where $(x(0), z(0)) \in \mathbb{R}^n \times \mathcal{V}$, $t \geq 0$.

Example:



$\mathcal{G}(\mathcal{V}, \mathcal{E})$

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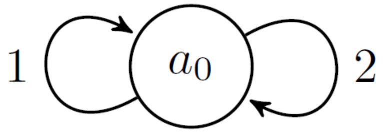
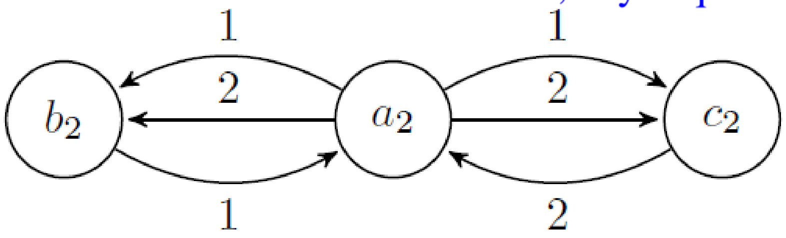
Path complete graph:

Example:

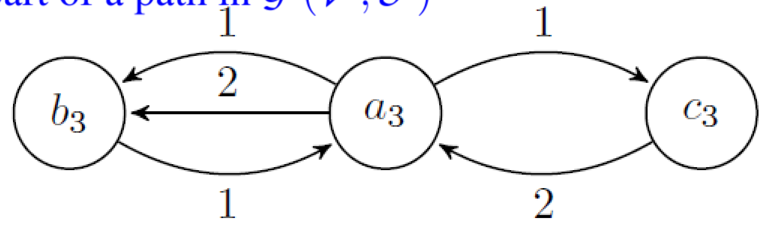


$\mathcal{G}(\mathcal{V}, \mathcal{E})$

i.e., any sequence σ in $\mathcal{G}(\mathcal{C}, \mathcal{E})$ can be realized as part of a path in $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$



Path-complete graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$

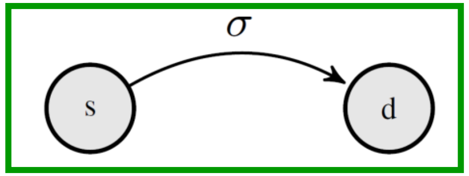


Not path-complete graph (222 cannot be generated)

Path Complete Lyapunov Functions

Idea: Describe the decrease conditions of Multiple LFs in a graph: unifies previous approaches

Path Complete LFs:



The set $V_i(\cdot) : \mathbb{R}^n \to \mathbb{R}_+, i = 1, ..., |\mathcal{V}'|$, is a Path-Complete Lyapunov Function if for all x, for all $(s, d, \sigma) \in \mathcal{E}'$

$\Longleftrightarrow$

$$V_d(A_\sigma x) \leq \varepsilon^{|\sigma|} V_s(x),$$

for some $\varepsilon \in [0, 1)$.

Path Complete Lyapunov Functions

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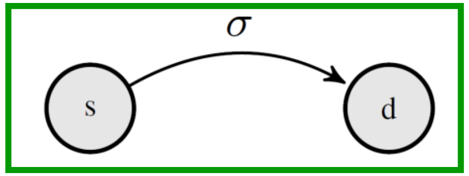
Path Complete LFs:



The set $V_i(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$, $i = 1, \ldots, |\mathcal{V}'|$, is a Path-Complete Lyapunov Function if for all x, for all $(s, d, \sigma) \in \mathcal{E}'$

$$V_d(A_\sigma x) \leq \varepsilon^{|\sigma|} V_s(x), \quad \text{for some } \varepsilon \in [0, 1).$$

Challenge: find a Lyapunov function / invariant set

-> simple yet rich enough template that leads to solvable decrease conditions

-> add complexity in the path complete graph structure

Path Complete Lyapunov Functions

Idea: Describe the decrease conditions of Multiple LFs in a graph: unifies previous approaches

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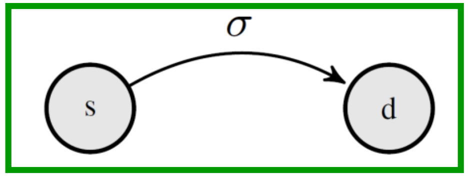

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Polyhedral functions

-> add complexity in the path complete graph

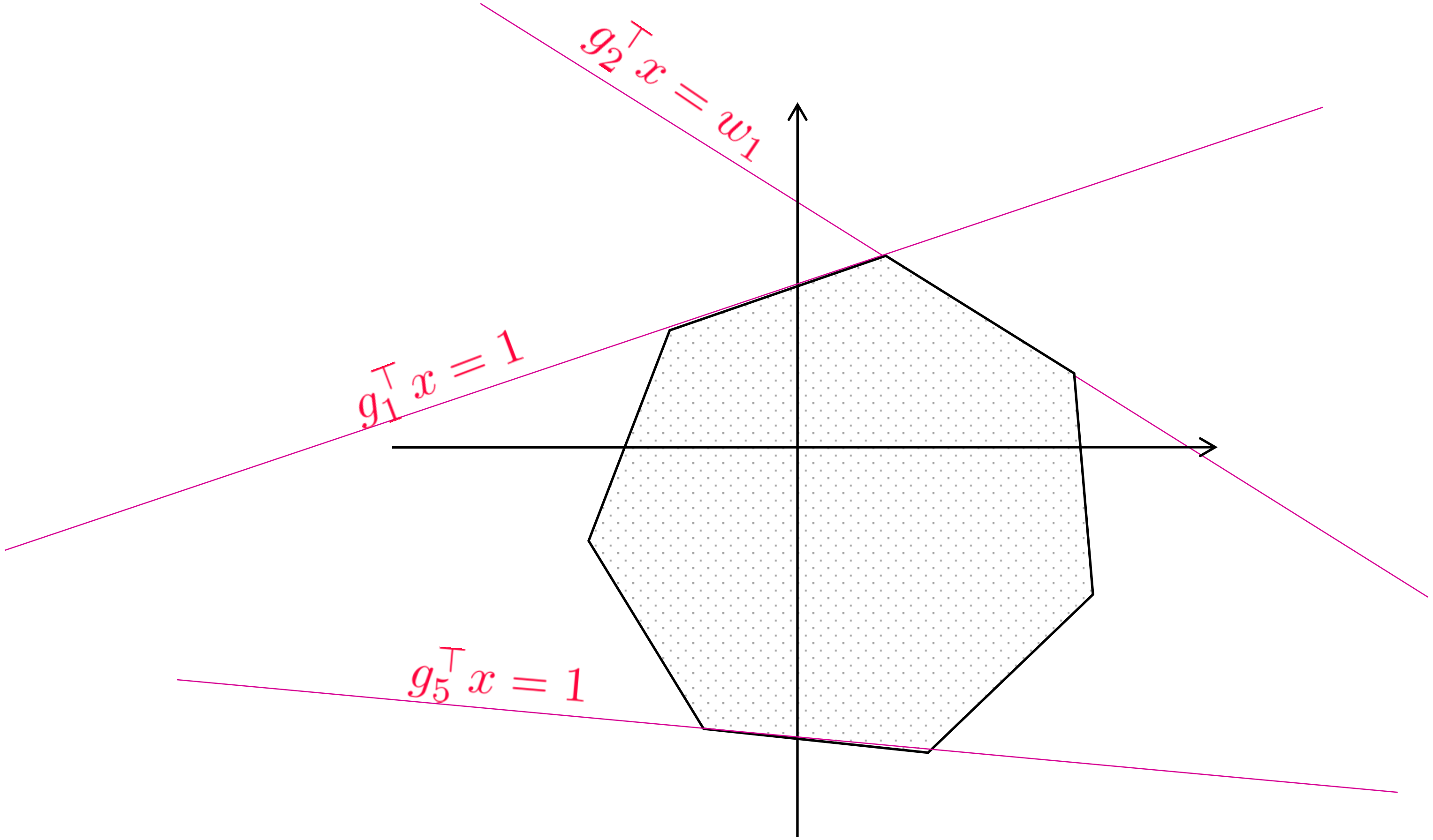
Partial Liftings

Polyhedral Path Complete Lyapunov Functions

Polyhedral sets

hyperplanes $\mathcal{S} = \{x : Gx \leq w\} = \{x : V(x) \leq 1\}$

$$V(x) = \max_{i=1,..,m} \left\{ \frac{(Gx)_i}{w_i} \right\}$$



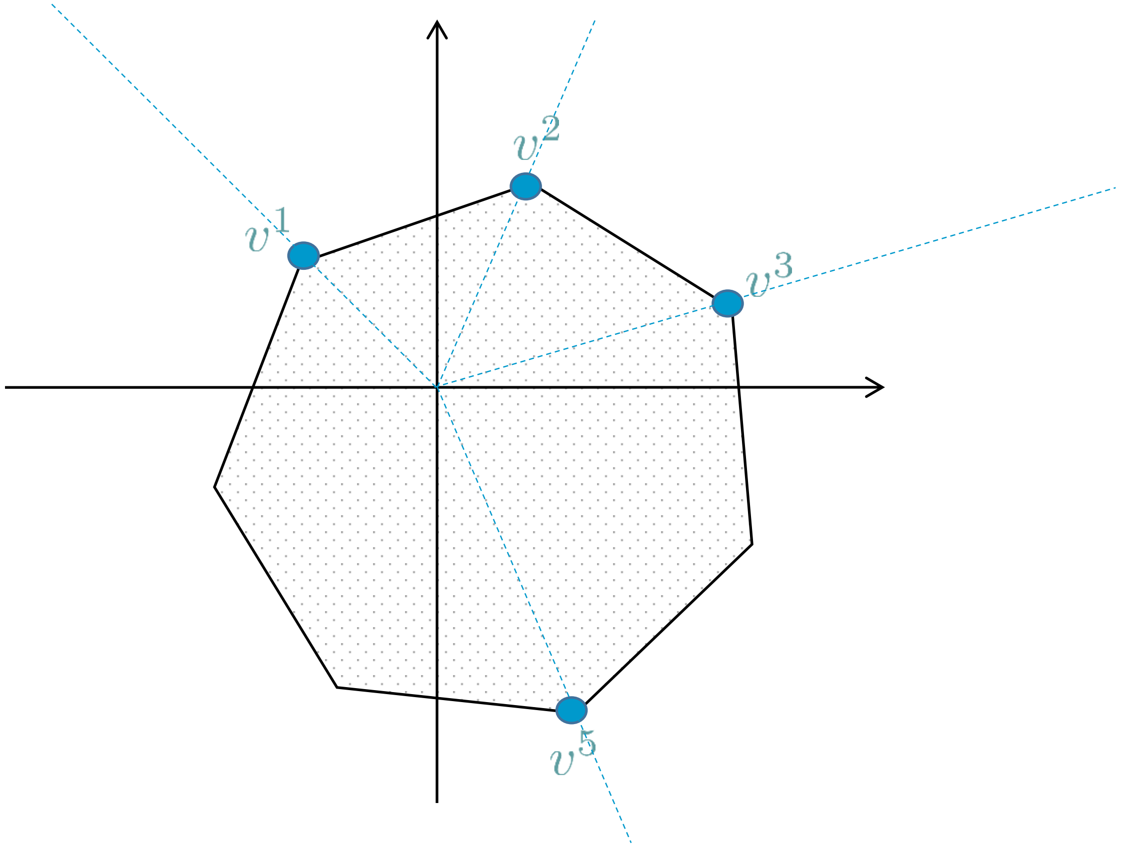
Polyhedral Path Complete Lyapunov Functions

Polyhedral sets

rays

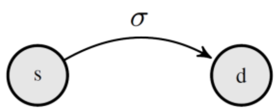
$$\mathcal{S} = \{Vy : c^\top y \leq 1, y \geq 0\}$$

$$V(x) = \min_{y \geq 0} \{c^\top y : x = Vy\}$$



Polyhedral Path Complete Lyapunov Functions

Condition (decrease/invariance conditions)



The following are equivalent.

(i) $V_d(A_\sigma x) \leq \varepsilon^{|\sigma|} V_s(x)$.

(ii) There exists a nonnegative matrix $H \in \mathbb{R}^{m_d \times m_s}$ such that $G_d A_\sigma = H G_s$ and $H w_s \leq \varepsilon^{|\sigma|} w_d$.

(iii) There exists a nonnegative matrix $P \in \mathbb{R}^{q_d \times q_s}$ such that $A_\sigma V_s = V_d P$ and $c_d^\top P \leq \varepsilon^{|\sigma|} c_s^\top$.

Polyhedral Path Complete Lyapunov Functions

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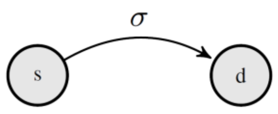

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Challenge: bilinear equations, (G,H) or (V,P) are unknown

Polyhedral Path Complete Lyapunov Functions

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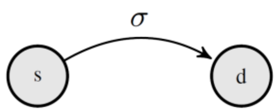

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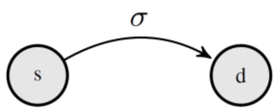
Solution: fix G (or V) and solve only for w (c)

Polyhedral Path Complete Lyapunov Functions

Condition (decrease/invariance conditions)



$$(i) \quad V_d(A_\sigma x) \leq \varepsilon^{|\sigma|} V_s(x).$$

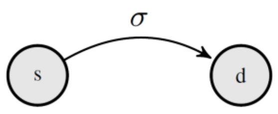
$$(ii) \quad |G_d A_\sigma G_s^{-1}| w_s \leq \varepsilon^{|\sigma|} w_d.$$

$$(iii) \quad c_d^\top |V_d^{-1} A_\sigma V_s| \leq \varepsilon^{|\sigma|} c_s^\top.$$

We fix the hyperplanes (rays) directions for polytopes with 2n parallel faces (opposite vertices)

Polyhedral Path Complete Lyapunov Functions

Condition (decrease/invariance conditions)



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We fix the hyperplanes (rays) directions for polytopes with 2n parallel faces (opposite vertices)

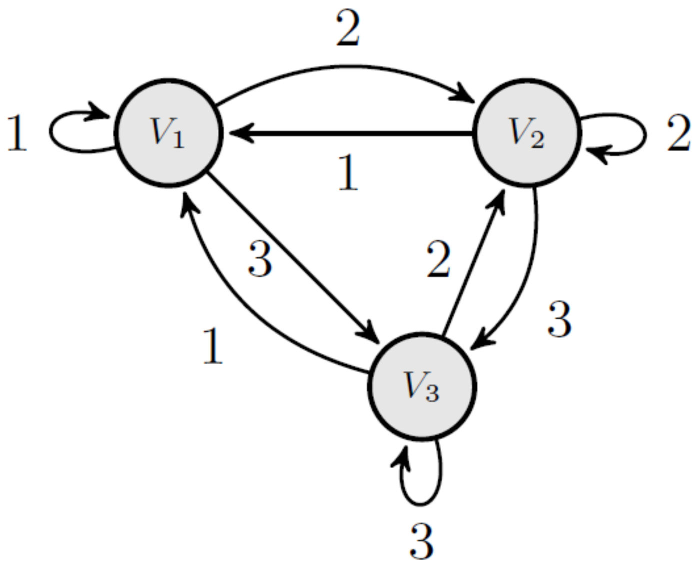
But how restrictive is this choice?

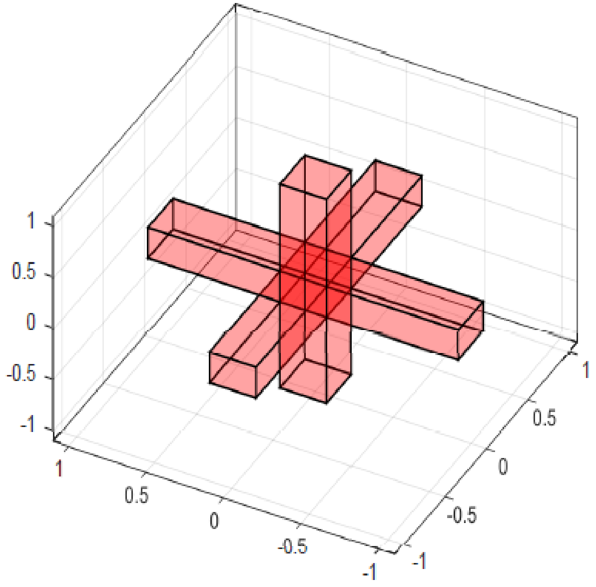
A motivating example

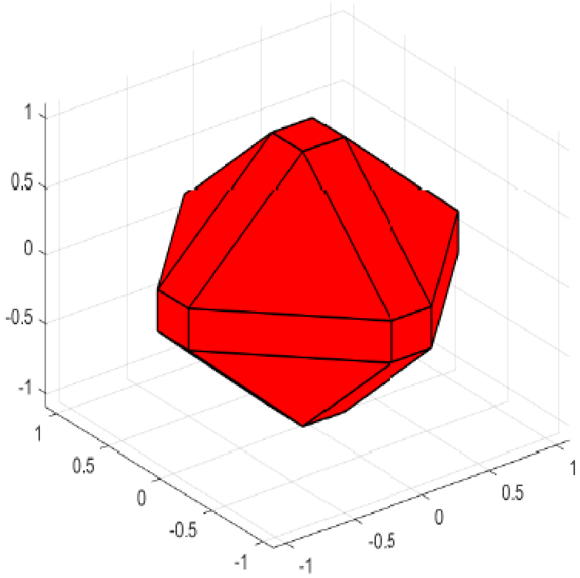
$$\mathcal{A}_1 = \varepsilon \left\{ \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ 1 & 1 & \dots & 1 \end{bmatrix} \right\}$$

No Lyapunov function can be found by weighting an infinity norm (any componentwise scaling of the hypercube)

However, a Path-Complete Lyapunov function with pieces weighted infinity norms can always be found, for a path-complete graph







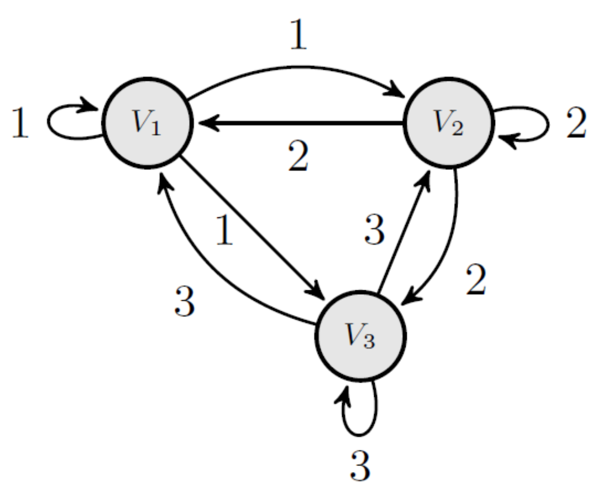
Intuition: The convex hull of the sublevel set of the path-complete Lyapunov function approximates the crosspolytope (which is a natural invariant set)

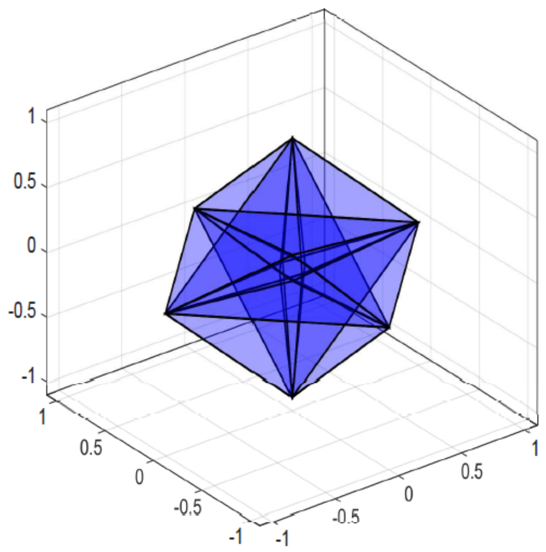
A motivating (dual) example

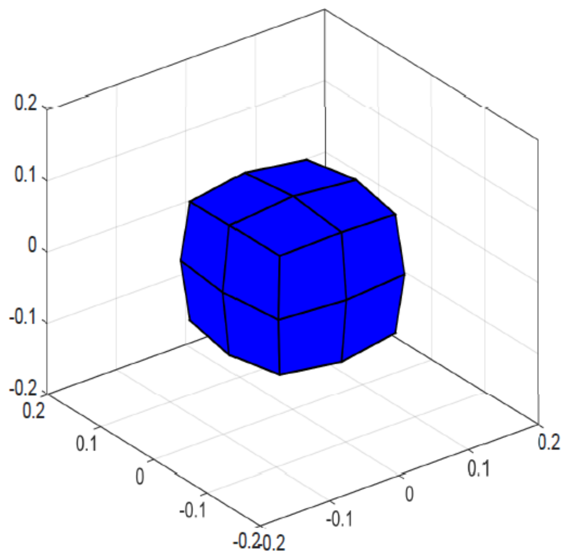
$$\mathcal{A}_2 = \varepsilon \left\{ \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ . & . & \dots & . \\ 1 & 0 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ . & . & \dots & . \\ 0 & 0 & \dots & 1 \end{bmatrix} \right\}$$

No Lyapunov function can be found by weighting a 1-norm (any componentwise scaling of the cross-polytope)

However, a Path-Complete Lyapunov function with pieces weighted 1-norms can always be found, for a path-complete graph







Intuition: The sublevel set of the path-complete Lyapunov function approximates the hypercube (which is a natural invariant set)

How to produce better path-complete graphs?

There are known liftings on path-complete graphs that provide *hierarchies* *that guarantee asymptotic convergence to stability certificates for switching systems, e.g.,*

- T/T*-Lifts; use iterated dynamics forwards/backwards in time by adding edges to the graph [Ahmadi, Jungers et al. 2014, Philippe, Jungers et al. 2016]

-P/P* -Lifts; distinguish between different possible switching sequences forwards/backwards in time by adding nodes to the graph [Lee and Dullerud 2006, Essick et al. 2014]

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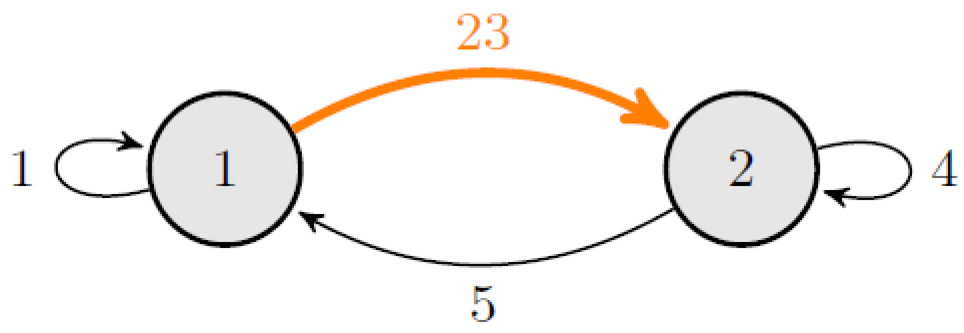
Challenge: target the lifting in parts of the path complete graph

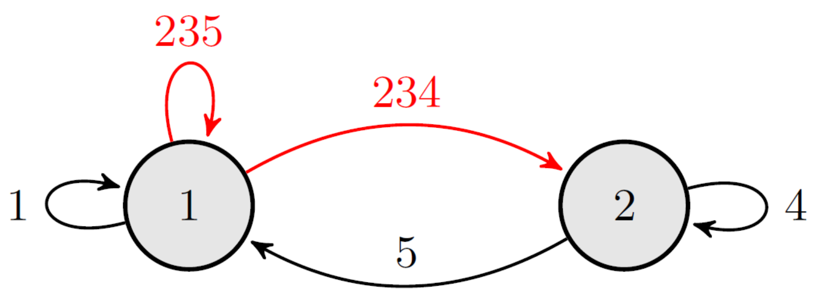
Partial lifts

Partial lifts

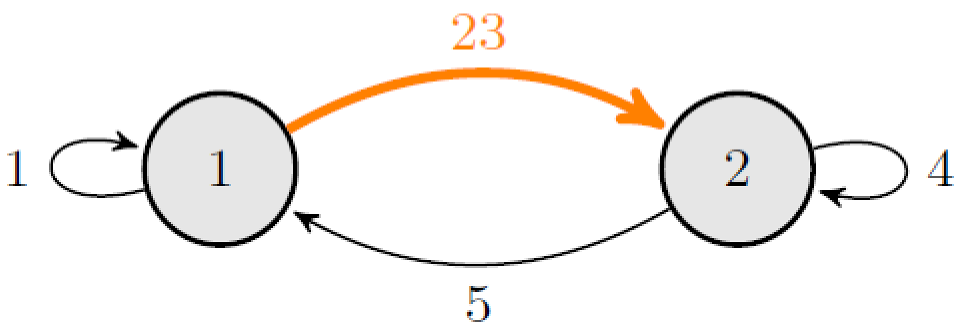


$\mathcal{G}(\mathcal{V},\mathcal{E}) = \mathcal{G}'(\mathcal{V}',\mathcal{E}')$

Partial lifts



Partial T-Lift



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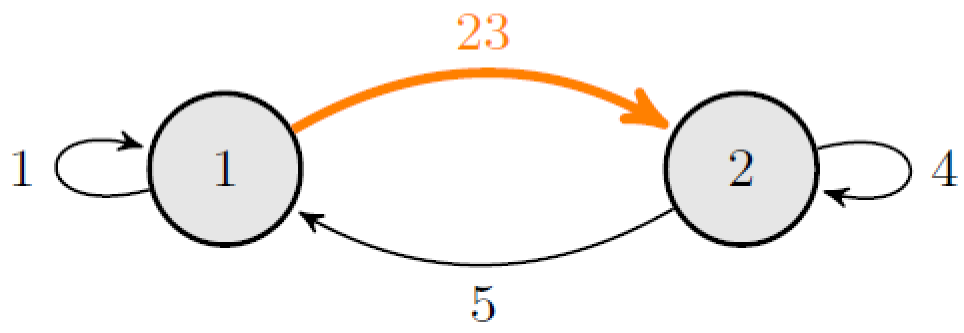

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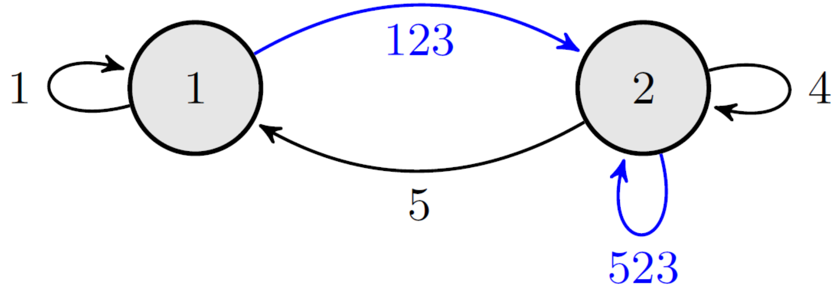

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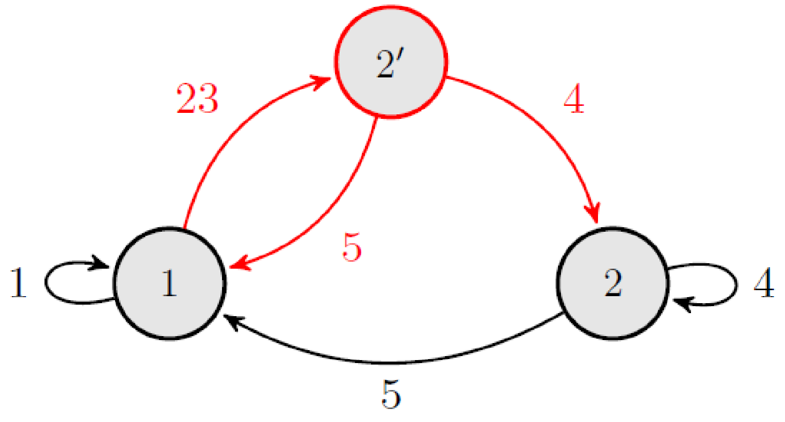
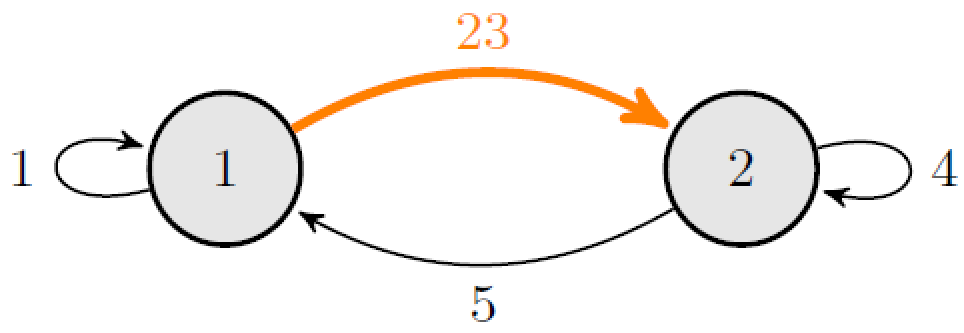
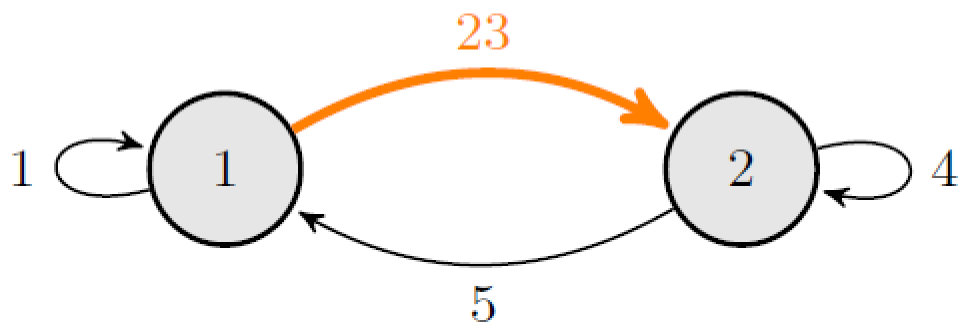


Partial P-Lift



$\mathcal{G}(\mathcal{V},\mathcal{E}) = \mathcal{G}'(\mathcal{V}',\mathcal{E}')$

Partial lifts



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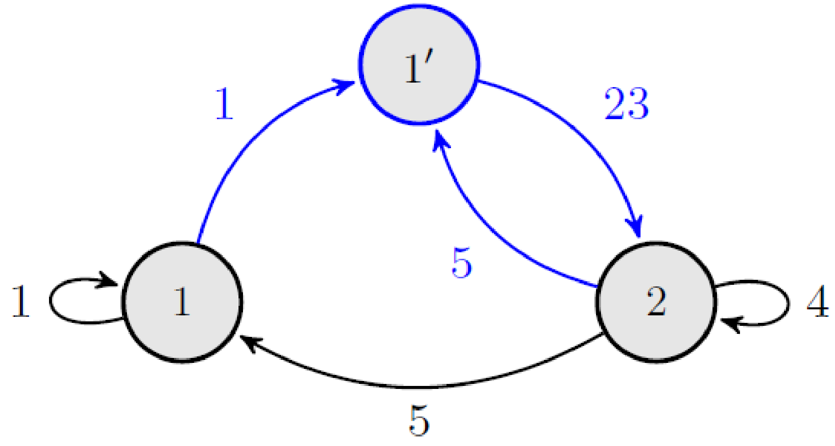
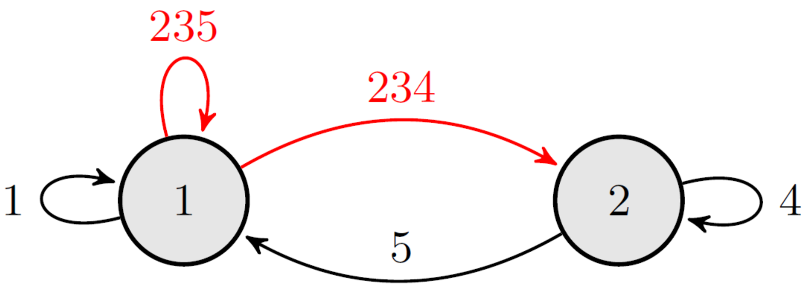
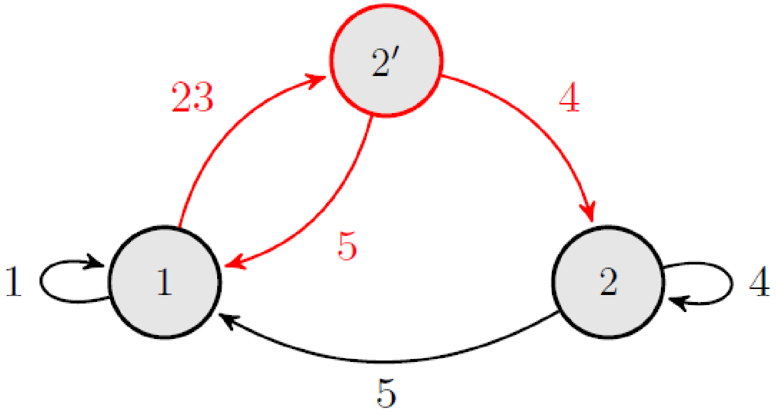


Partial P*-Lift

Partial lifts



Partial T-Lift



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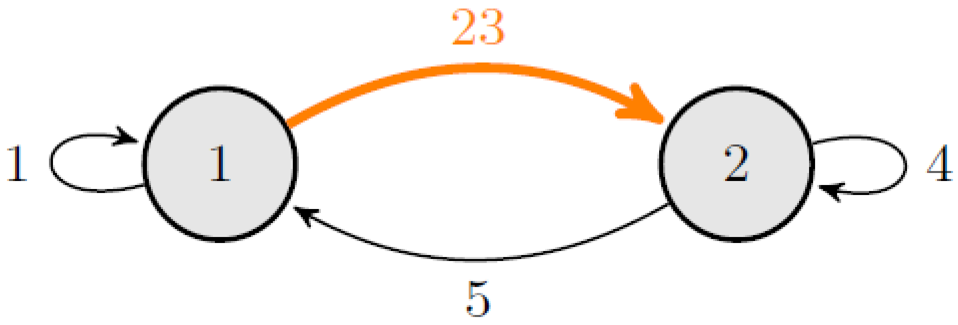

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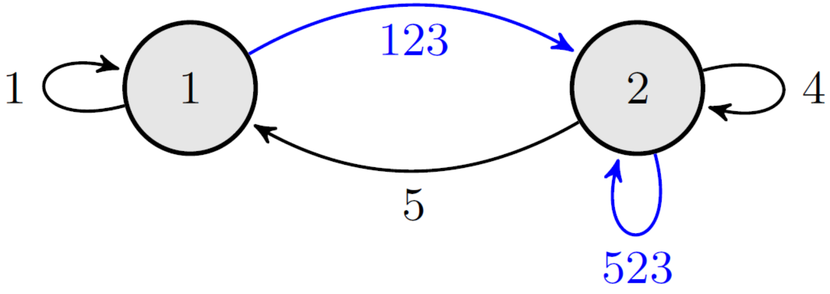

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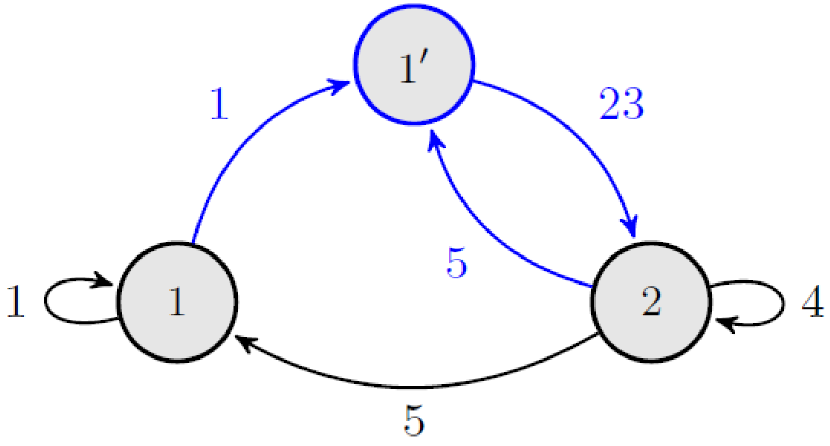
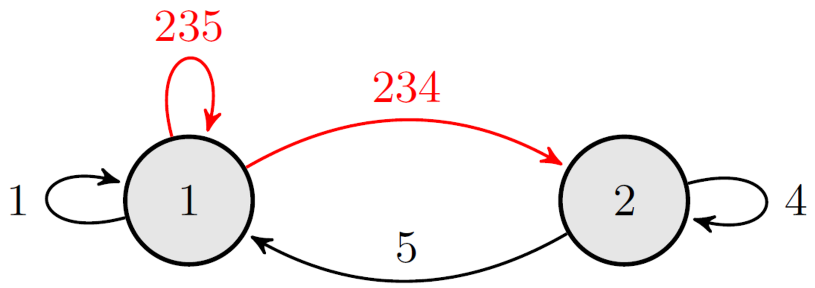


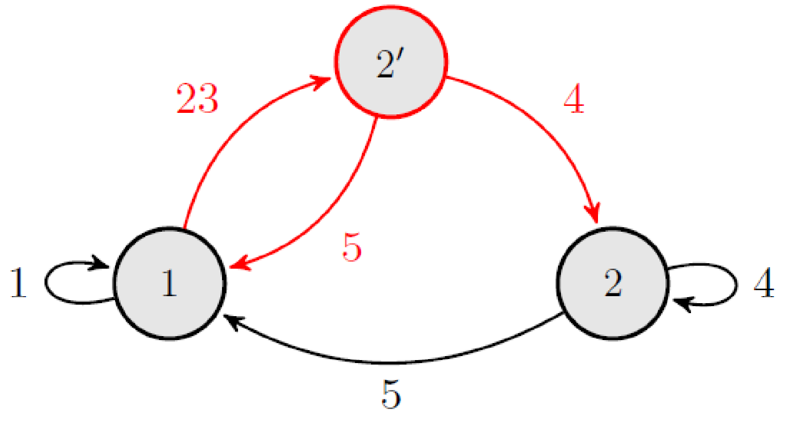
Partial P*-Lift

Partial lifts

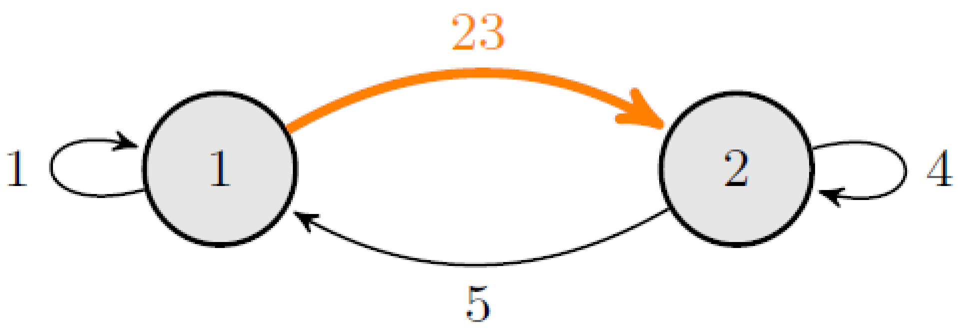


Partial T-Lift

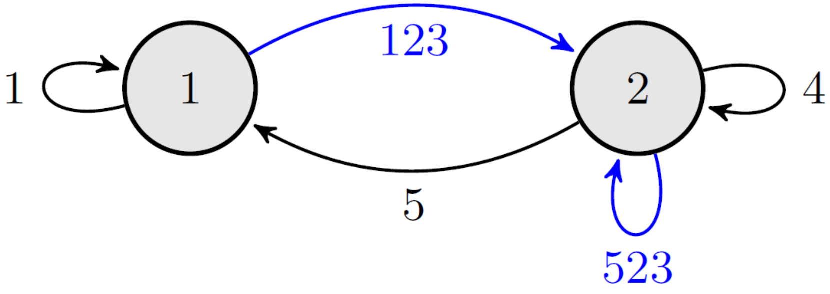
All four operations preserve path-completeness



Partial P-Lift

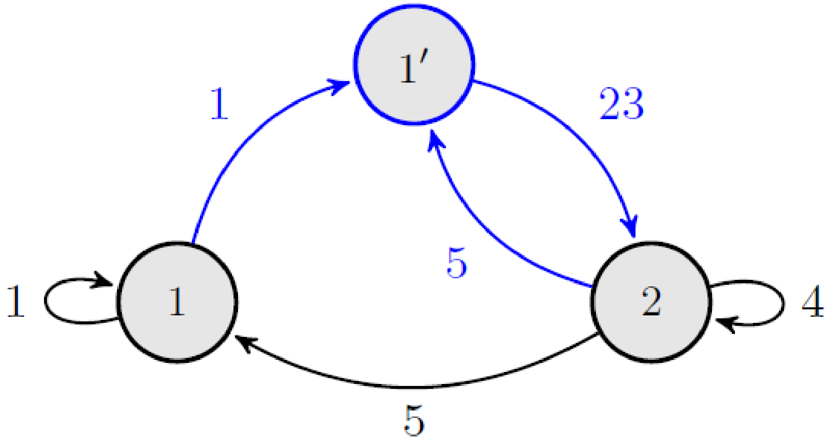


$\mathcal{G}(\mathcal{V},\mathcal{E}) = \mathcal{G}'(\mathcal{V}',\mathcal{E}')$



Partial T*-Lift

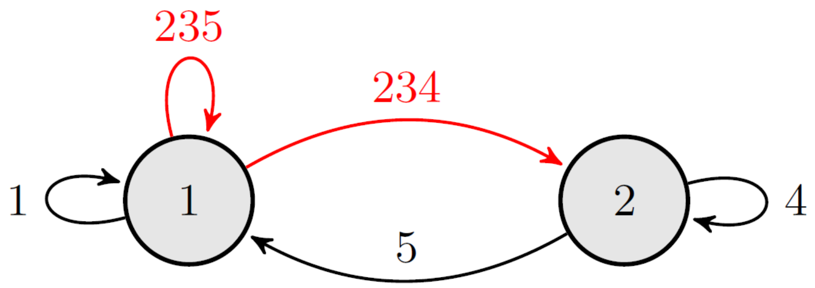
We can formulate algorithms to target the liftings



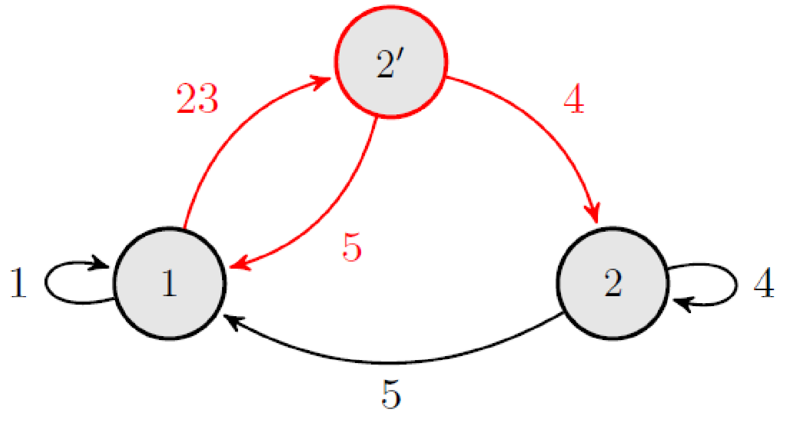
Partial P*-Lift

How to choose which edge to lift?

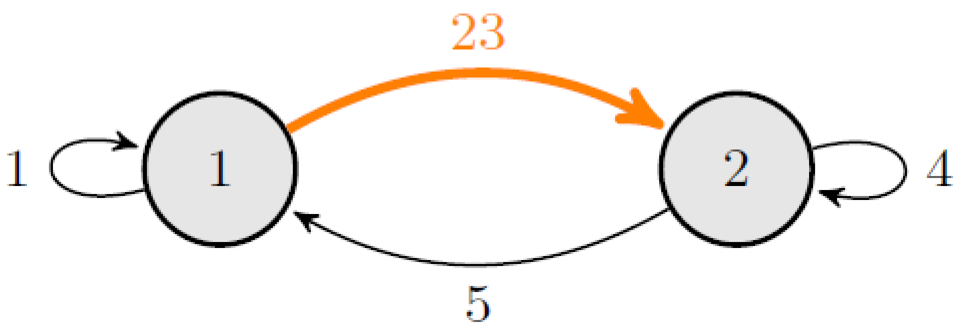
Intuition: Lift edges corresponding to conditions that violate the decrease inequality



Partial T-Lift



Partial P-Lift



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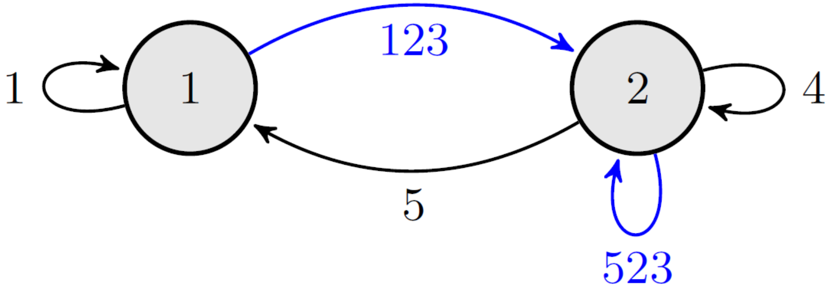
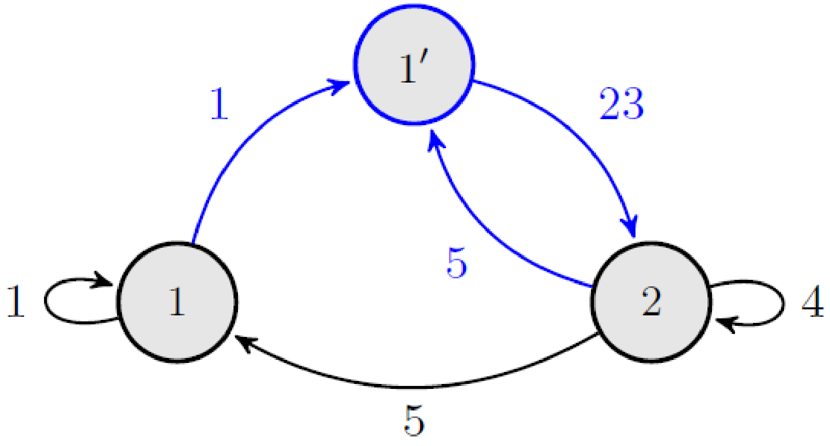


Partial T*-Lift



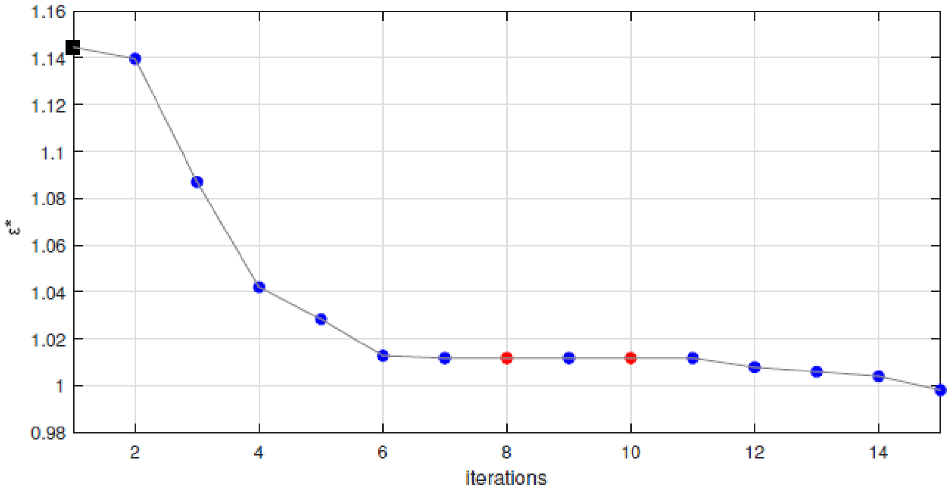
Partial P*-Lift

Example

$$x(t+1) = A_{\sigma(t)}(t), \sigma(t) \in \{1, 2\}$$

$$A_1 = \begin{bmatrix} 0.095 & -0.2375 & 0.2375 \\ -1.9 & 0 & 0 \\ 0 & 0 & 0.475 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.4753 & 0 & 0 \\ 0 & 0 & 0.4753 \\ 0 & -1.9012 & 0 \end{bmatrix},$$



$$\varepsilon^* = min\{\varepsilon : V_d(A_\sigma x) \leq \varepsilon V_s(x), (i, j, \sigma) \in \mathcal{G}'(\mathcal{V}', \mathcal{E}')\}$$

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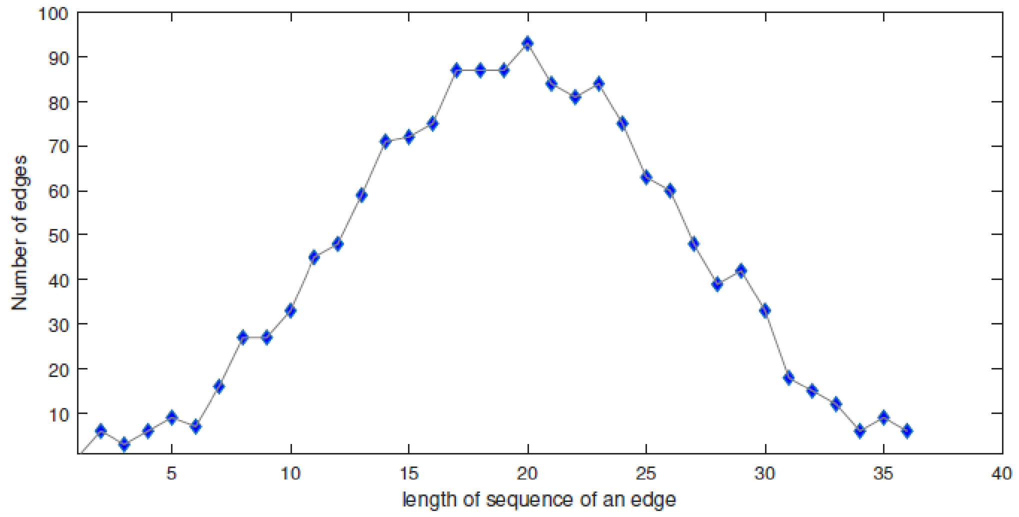

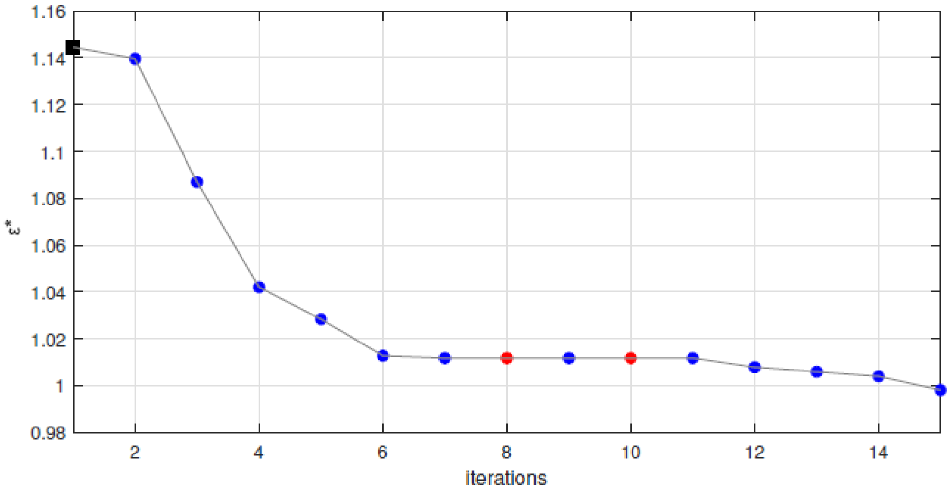

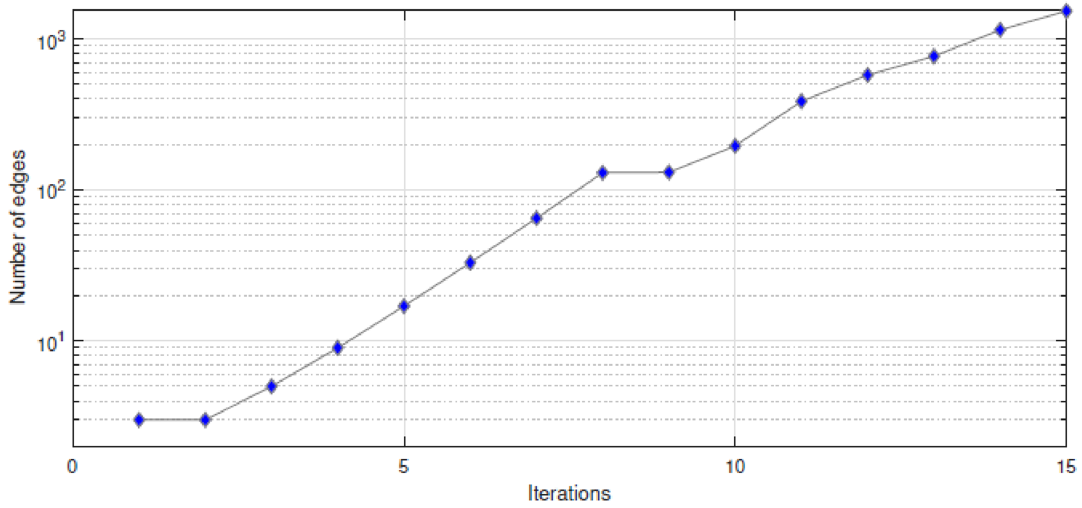

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Conclusions

- Polyhedral Path Complete Lyapunov functions -> solvable algebraic conditions
- Partial liftings can be used instead of full liftings, tradeoff between conservatism and practicality
- LP-based algorithms can be formulated to obtain less conservative stability/invariance conditions

Future:

- Hierarchies of stability certificates
- Other templates
- Control problem

Questions / Suggestions?

Thank you!

Path Complete Lyapunov Functions

Constrained Switching Systems:

Let $\mathcal{A} = \{A_1, ..., A_N\}$, a labeled directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$

$$x(t+1) = A_{\sigma(t)} x(t)$$
$$z(t+1) \in \mathrm{Out}(z(t), \mathcal{G}(\mathcal{V}, \mathcal{E}))$$
$$(z(t), z(t+1), \sigma(t)) \in \mathcal{E}$$

where $(x(0), z(0)) \in \mathbb{R}^n \times \mathcal{V}$, $t \geq 0$.

Path complete graph:

$\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ is *path-complete* if for an admissible $\sigma = \sigma_1 \sigma_2 ... \sigma_k$ in a path in $\mathcal{G}(\mathcal{V}, \mathcal{E})$, $k \geq 1$, there are two nodes $s \in \mathcal{V}', d \in \mathcal{V}'$, a sequence $\sigma^\star = \sigma_1 \sigma \sigma_2$, such that $\sigma^\star \in \sigma(s, d)$

i.e., any sequence σ in $\mathcal{G}(\mathcal{C}, \mathcal{E})$ can be realized as part of a path in $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$