Deciding Reachability for Piecewise Constant Derivative Systems on Orientable Manifolds

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Reachability Problems'19

Motivation

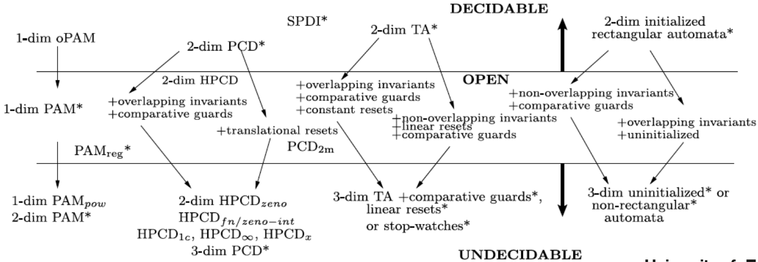
Understanding the boundary between decidable and undecidable hybrid systems

A **hybrid automaton** is

(a) a finite state machine combined with some k real-valued continuous variables

(b) k determines the number of the automaton dimensions

Asarin, Mysore, Pnueli and Schneider, 2011
'Low dimensional hybrid systems - decidable, undecidable, don't know':

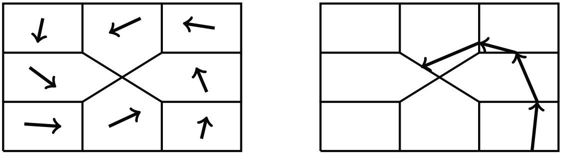


A Piecewise Constant Derivative System (PCDs) is a finite set of regions, where each region is associated with a constant vector field.

A 2-dimensional Piecewise Constant Derivative System (or 2-PCD) is a pair $\mathsf{H} = (\mathsf{Q}, \mathsf{F})$ with $\mathsf{Q} = \{q_i\}_{i \in I}$ a finite polygonal partition of $\mathbb{R}^2$ and $\mathsf{F} = \{\vec{v}_i\}_{i \in I}$ a set of vectors from $\mathbb{R}^2$. The dynamics is determined by the equation $\dot{x} = \vec{v}_i$ for $x \in q_i$.

Example

An example of a 2-PCD and a trajectory segment:



Reachability Problem for PCDs

Point-to-point reachability:

A point b is reachable from a point a if there is a trajectory segment that starts at a and ends at b.

Edge-to-edge reachability:

An edge e_f is reachable from an edge e_s if there are points $a_s \in e_s$ and $b_f \in e_f$ such that there is a trajectory segment that starts at a_s and ends at b_f.

Decidability:

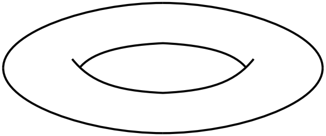
(Maler and Pnueli, 1993) Decidable for the 2-dimensional case: Each trajectory is ultimately periodic

(Asarin, Maler and Pnueli, 1995) Undecidable for 3-dimensions

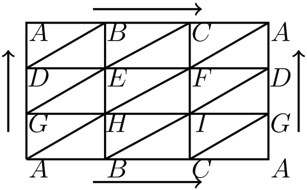
Representations of a torus:

(a) A surface in $\mathbb{R}^3$
(b) A triangulated surface with identified edges

A closed surface is a compact surface without boundary



(a)



(b)

PCD on a 2-dimensional manifold (PCD_{2m}) is a 2-PCD on a closed orientable surface S.

Regular PCDs on Manifolds (PCD$_{r2m}$)

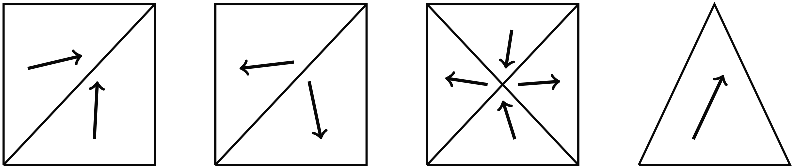
We forbid the following dynamics in PCD$_{r2m}$:

(a) collision at an edge;

(b) branching at an edge;

(c) collision and branching on a vertex;

(d) flow vector parallel to an edge



(a) (b) (c) (d)

Regionwise Dynamical Systems (RDS_{2m})

A. Mayer: Trajectories on the closed orientable surfaces, 1943

A finite number of regions r_i, homeomorphic to a Euclidean disc with he dynamics in each r_i defined by :

$$\varphi_i' = \Phi(\varphi_i, \psi_i), \quad \psi_i' = \Psi(\varphi_i, \psi_i)$$

A trajectory τ is dense on a set of intervals $e_1, \ldots, e_k$ if for any $x \in \tau$ and any interval $e_i' \subseteq e_i$, $1 \leqslant i \leqslant k$, there is $y \in e_i'$ such that y is reachable from x.

A trajectory τ is called orbital stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if a trajectory τ' starts in the δ-neighbourhood of τ then it is also contained in the ε-neighbourhood of τ.

Properties of Trajectories of RDS_{2m}

Any RDS_{2m} can be decomposed into components consisting of trajectories which are equivalent topologically.

Theorem (Mayer):

A RDS_{2m} S_g is a disjoint union of a finite number of areas of the following types:

Type A: Any trajectory inside the area is orbital stable and non-closed (**the area is flat!**).

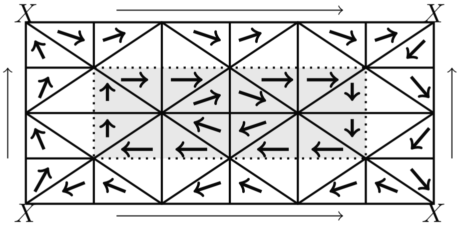
Type B: Any trajectory inside the area is closed

Type C: Any trajectory inside the area is everywhere dense

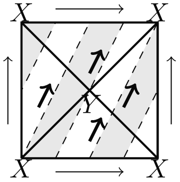
All other trajectories, called **separatrices**, form boundaries between the areas of the above types.

Examples

The component is of Type A is highlighted with grey colour:



A PCD_{r2m} consisting of one dynamical component of Type B:



Language of PCD_{r2m}

A word $e_1 \dots e_k$ is a sequence of edges traversed by some trajectory segment.

A language L is called **uniformly recurrent** if for any $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that every word from L of length k contains all of the words from L of length n as subwords.

Lemma.
Density of the trajectories of a dynamical component of Type C of a PCD_{r2m} $\mathsf{H} = (\mathsf{Q}, \mathsf{F})$ implies the uniformal recurrence of its language L_C.

Corollary.
The language of any dynamical component of Type B of any PCD_{r2m} is uniformly recurrent.

Rauzy Graphs

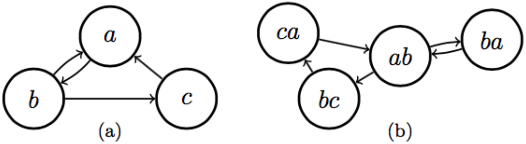
Rauzy graph of power $k \geqslant 1$ for a language L is a directed graph $R^k(L) = (V^k, E^k)$ defined as follows:

(1) $V^k = \{w \in L \mid |w| = k\}$;

(2) For any two vertices $u = u_1 u_2 \dots u_k \in V^k$ and $v = v_1 v_2 \dots v_k \in V^k$ there is an edge $(u, v) \in E^k$ if $u_2 = v_1, u_3 = v_2, \dots, u_k = v_{k-1}$ and $u_1 u_2 \dots u_k v_k \in L$.

Example

The Rauzy graphs of power one and two for $L = \{a, b, c, ab, bc, ba, ca, abc, aba, bab, bca, cab\}$:



Decidability of Edge-to-Edge Reachability for PCD$_{r2m}$

<u>Theorem.</u>

Edge-to-edge reachability for a PCD$_{r2m}$ is decidable.

Idea of the proof:

For any PCD$_{r2m}$ $\mathsf{H} = (\mathsf{Q}, \mathsf{F})$, we can construct a sequence of Rauzy graphs such that the set V^k of vertices of $R^k(L) = (V^k, E^k)$ consists of all the words of length k over the finite alphabet on the edges of H.

This set can be constructed by applying the successor function $k - 1$ times to each edge.

There is a finite t_{stop} such that an edge $e_f \in \mathrm{e}(\mathsf{Q})$ is reachable from an edge e_s if and only if $R^{t_{stop}}(L)$ contains a component with a vertex labelled by a word $(\dots e_s \dots e_f \dots)$.

Future Work

While the reachability problem for the whole class PCD_{2m} is still an open question, we proved that under certain limitations on the systems dynamics it becomes decidable.

As future work we consider to extend the current results to non-orentable manifolds using properties of trajectories on non-orientable manifolds presented in

Aranson, S.H.: Trajectories on non-orientable two-dimensional manifolds.

Thank you