Solving Word Equations Using SAT

Joel D. Day, Thorsten Ehlers, **Mitja Kulczynski**, Florin Manea,
Dirk Nowotka and Danny B. Poulsen

RP ’19

The Basics

These are word equations

a
b

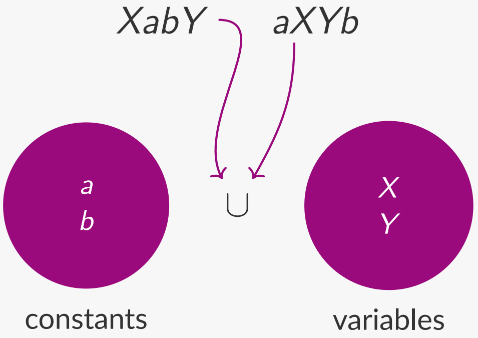
constants

X
Y

variables

The Basics

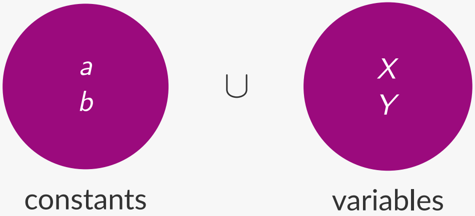
These are word equations



The Basics

These are word equations

$$XabY \doteq aXYb$$



The Basics

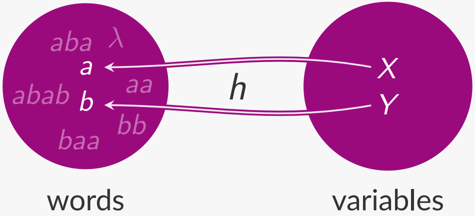
These are word equations

$$XabY \doteq aXYb$$

The Basics

These are word equations

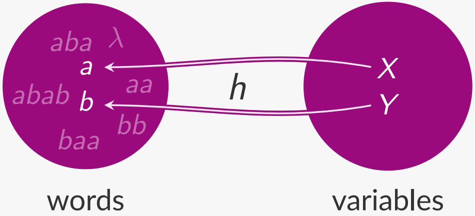
$$XabY \doteq aXYb$$



The Basics

These are word equations

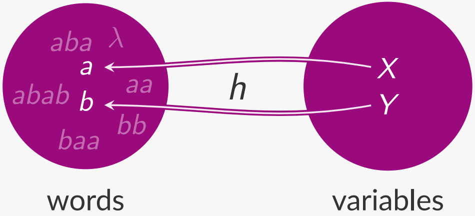
$$a\ ab\ b \doteq a\ a\ b\ b$$



The Basics

These are word equations

$$XabY \doteq aXYb$$
$$aYXYXaaaYb \doteq XbXYaXaXbb$$



The Basics

These are bounded word equations

$$XabY \doteq aXYb$$

with bounds $b_X = b_Y = 1$.

The Basics

These are bounded word equations

$$XabY \doteq aXYb$$

with bounds $b_X = b_Y = 1$.

The Basics

These are bounded word equations

$$XabY \doteq aXYb$$

with bounds $b_X = b_Y = 1$.

words

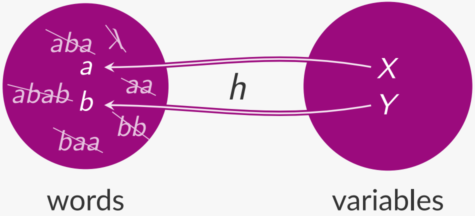
variables

The Basics

These are bounded word equations

$$XabY \doteq aXYb$$

with bounds $b_X = b_Y = 1$.

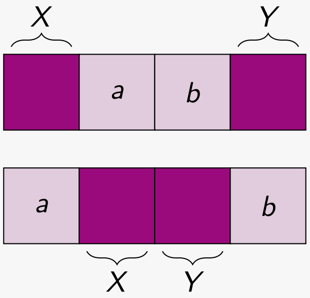


The Basics

These are bounded word equations

$$XabY \doteq aXYb$$

with bounds $b_X = b_Y = 1$.

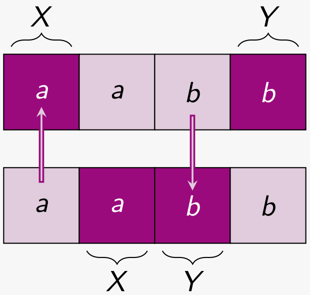


The Basics

These are bounded word equations

$$XabY \doteq aXYb$$

with bounds $b_X = b_Y = 1$.



The Idea

How this idea came to our minds

Solve word equations by guessing a bound for each variable.

The Idea

How this idea came to our minds

Solve word equations by guessing a bound for each variable.

easily solvable by using the *filling the positions method*.

The Idea

Solve word equations by guessing a bound for each variable.

easily solvable by using the *filling the positions method*.

… but finding these bounds is especially hard for balanced equations.

The Idea

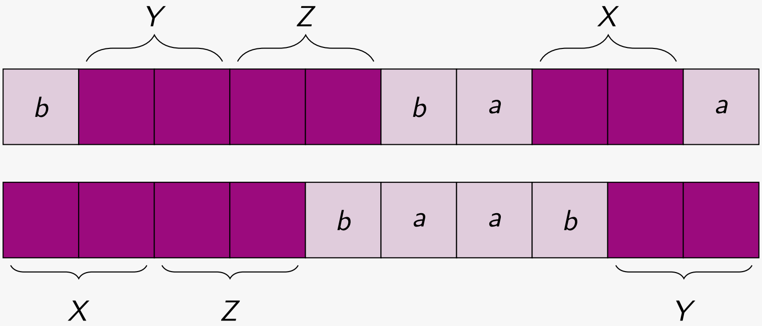
How this idea came to our minds

$$bYZbaXa \doteq XZbaabY$$

with bounds $b_X = b_Y = b_Z = 2$.

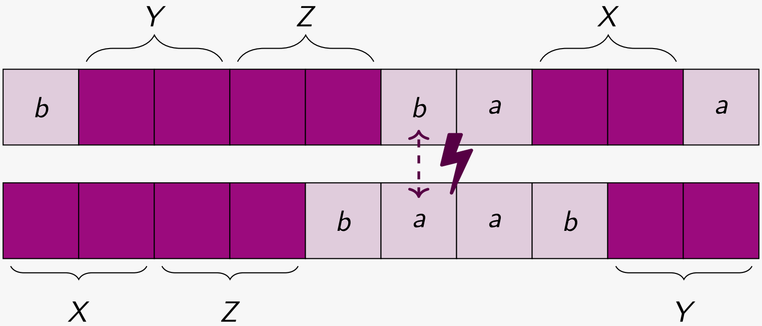
The Idea

How this idea came to our minds



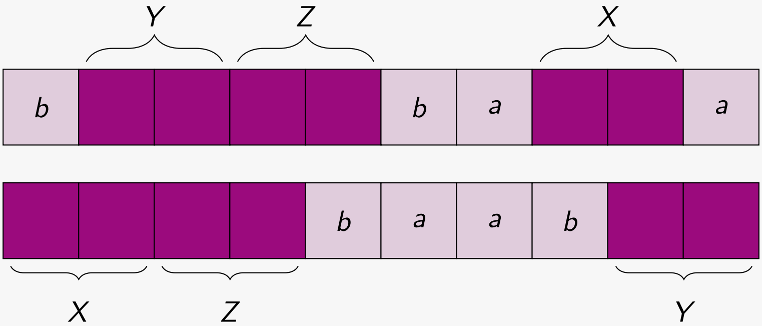
The Idea

How this idea came to our minds



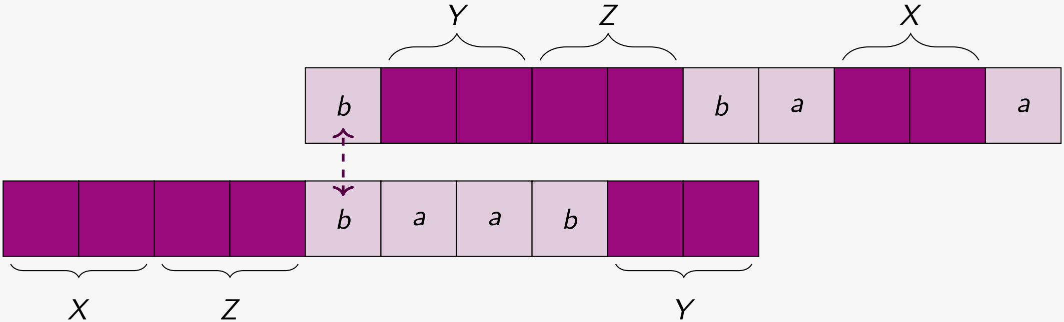
The Idea

How this idea came to our minds



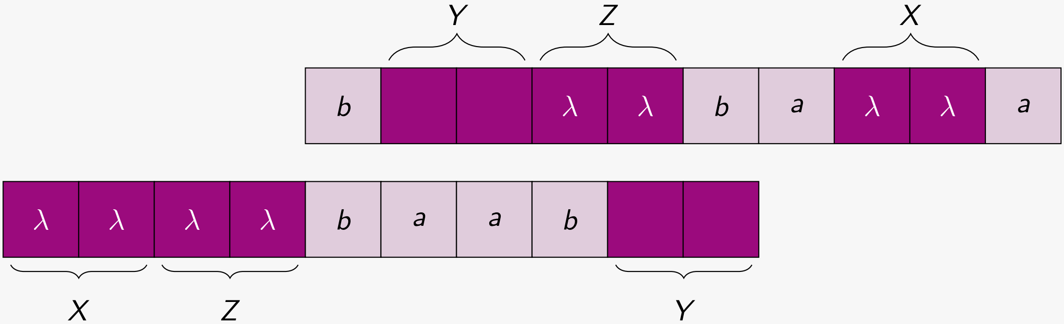
The Idea

How this idea came to our minds



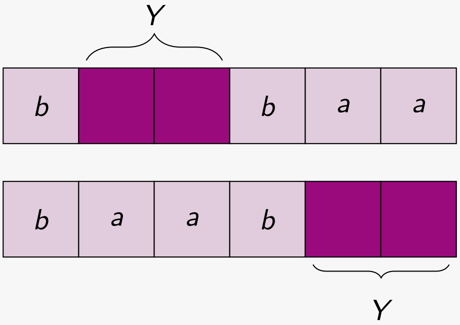
The Idea

How this idea came to our minds



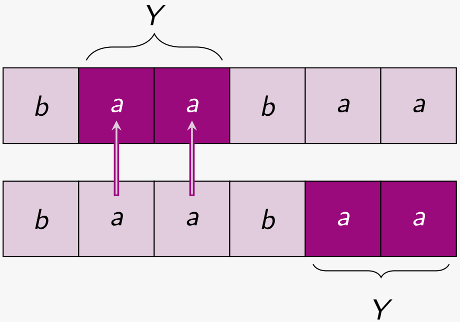
The Idea

How this idea came to our minds



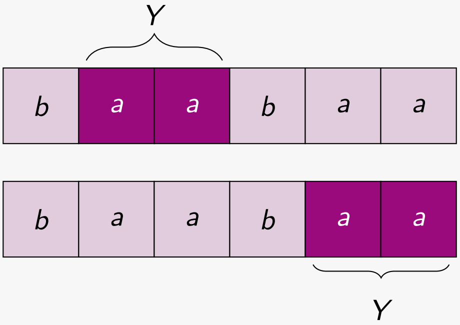
The Idea

How this idea came to our minds



The Idea

How this idea came to our minds



Gives us the substitution $X \mapsto \lambda$, $Y \mapsto aa$, $Z \mapsto \lambda$

The Idea

Build an automaton which mimics this behavior.

The Automaton

Brief overview about it's behaviour

$$aZXb \doteq aXaY$$

with bounds $b_X = b_Y = b_Z = 1$.

The Automaton

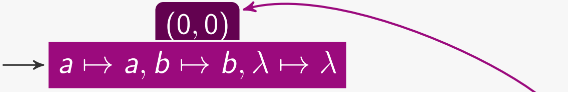
Brief overview about it's behaviour

$(0,0)$

$\longrightarrow$ $a \mapsto a, b \mapsto b, \lambda \mapsto \lambda$

The Automaton

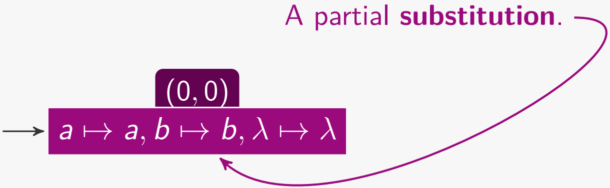
Brief overview about it's behaviour



A **location**, corresponding to the current position in our equation.

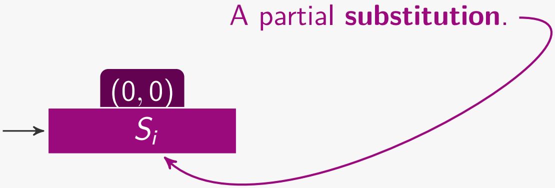
The Automaton

Brief overview about it’s behaviour



The Automaton

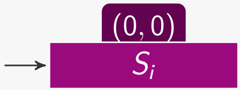
Brief overview about it's behaviour



The Automaton

Brief overview about it's behaviour

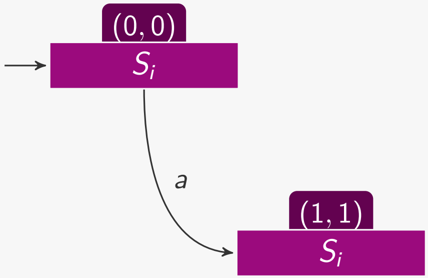
$$aZXb \doteq aXaY$$



The Automaton

Brief overview about it's behaviour

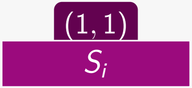
$$a\,ZXb \doteq a\,XaY$$



The Automaton

Brief overview about it's behaviour

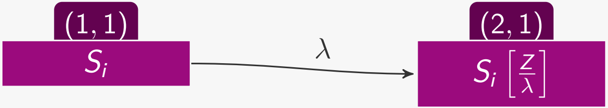
$$a\,ZXb \doteq a\,XaY$$



The Automaton

Brief overview about it's behaviour

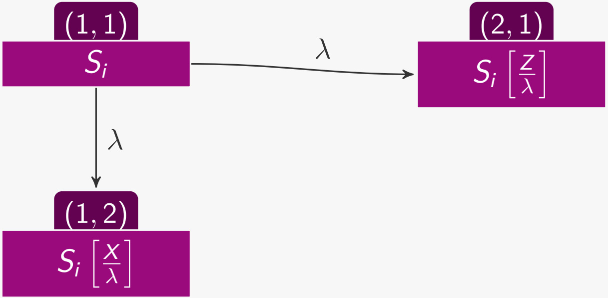
$$a\, ZXb \doteq a\, XaY$$



The Automaton

Brief overview about it's behaviour

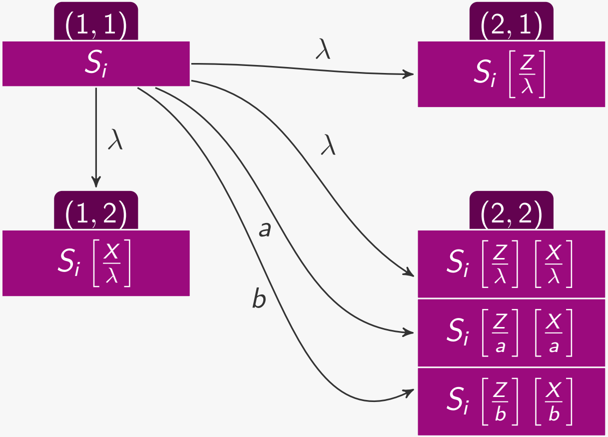
$$a\,ZXb \doteq a\,XaY$$



The Automaton

Brief overview about it's behaviour

$$a\, ZXb \doteq a\, XaY$$



The Automaton

Brief overview about it's behaviour

$$aZ\,Xb \doteq aX\,aY$$

$(2,2)$

$$S_i \left[\frac{Z}{\lambda}\right] \left[\frac{X}{\lambda}\right]$$

$$S_i \left[\frac{Z}{a}\right] \left[\frac{X}{a}\right]$$

$$S_i \left[\frac{Z}{b}\right] \left[\frac{X}{b}\right]$$

The Automaton

Brief overview about it’s behaviour

$$aZ\,Xb \doteq aX\,aY$$

The Automaton

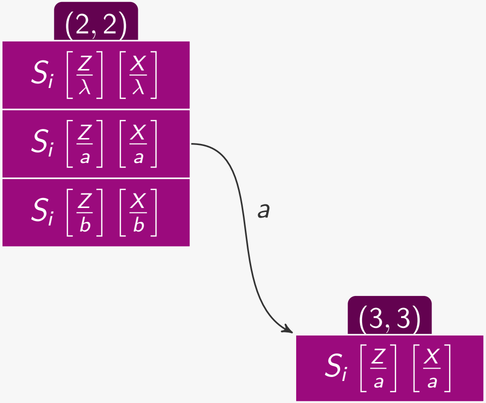
Brief overview about it’s behaviour

$$aZ\,Xb \doteq aX\,aY$$

The Automaton

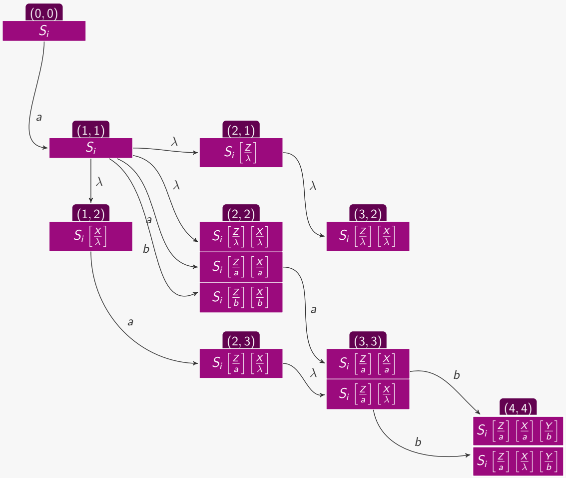
Brief overview about it's behaviour

$$aZ\,Xb \doteq aX\,aY$$



The Automaton

Brief overview about it's behaviour



The Automaton

Brief overview about it’s behaviour

$$aZXb \doteq aXaY$$

(4, 4)

$$S_i \left[\frac{Z}{a}\right] \left[\frac{X}{a}\right] \left[\frac{Y}{b}\right]$$

$$S_i \left[\frac{Z}{a}\right] \left[\frac{X}{\lambda}\right] \left[\frac{Y}{b}\right]$$

The Automaton

Brief overview about it's behaviour

$$aZXb \doteq aXaY$$

(4, 4)

$$S_i \begin{bmatrix} Z \\ \hline a \end{bmatrix} \begin{bmatrix} X \\ \hline a \end{bmatrix} \begin{bmatrix} Y \\ \hline b \end{bmatrix}$$

$$S_i \begin{bmatrix} Z \\ \hline a \end{bmatrix} \begin{bmatrix} X \\ \hline \lambda \end{bmatrix} \begin{bmatrix} Y \\ \hline b \end{bmatrix}$$

1. $X \mapsto a, Y \mapsto b, Z \mapsto a$
2. $X \mapsto \lambda, Y \mapsto b, Z \mapsto a$

Expansions

- Bound refinement to lower encoding complexity
- Search guiding by using Reduced Ordered Multi-Decision Diagrams (MDD)
- Linear length constraints by using MDDs

Speed up the search

$$aZXb \doteq aXaY$$

with bounds $b_X = b_Y = b_Z = 2$.

Speed up the search

$$u = aZXb \doteq aXaY = v$$
with bounds $b_X = b_Y = b_Z = 2$.

Build a Diophantine equation as follows:

$$\sum_{M \in \Gamma} (|u|_M - |v|_M) \cdot I_M = \sum_{a \in \Sigma} |v|_a - |u|_a$$

Speed up the search

$$u = aZXb \doteq aXaY = v$$

with bounds $b_X = b_Y = b_Z = 2$.

Build a Diophantine equation as follows:

$$\sum_{M \in \Gamma} (|u|_M - |v|_M) \cdot I_M = \sum_{a \in \Sigma} |v|_a - |u|_a$$

Gives the count of a's in u.

Speed up the search

$$u = aZXb \doteq aXaY = v$$

with bounds $b_X = b_Y = b_Z = 2$.

Build a Diophantine equation as follows:

$$\sum_{M \in \Gamma} (|u|_M - |v|_M) \cdot I_M = \sum_{a \in \Sigma} |v|_a - |u|_a$$

I_M are positive integer variables ranging within $\{ 0, \ldots, b_M \}$.

Speed up the search

$$u = aZXb \doteq aXaY = v$$

with bounds $b_X = b_Y = b_Z = 2.$

Build a Diophantine equation as follows:

$$\sum_{M \in \Gamma} (|u|_M - |v|_M) \cdot I_M = |v|_a - |u|_a + |v|_b - |u|_b$$

Speed up the search

$$u = aZXb \doteq aXaY = v$$

with bounds $b_X = b_Y = b_Z = 2$.

Build a Diophantine equation as follows:

$$\sum_{M \in \Gamma} (|u|_M - |v|_M) \cdot l_M = 2 - 1 + 0 - 1 = 0$$

Speed up the search

$$u = aZXb \doteq aXaY = v$$

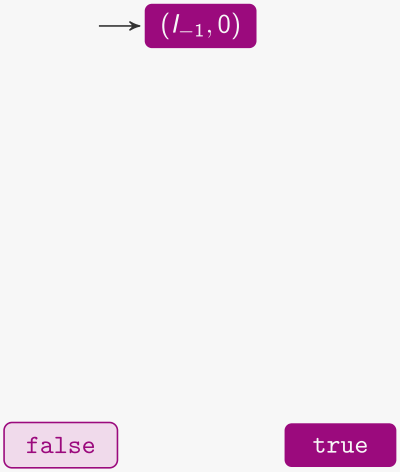
with bounds $b_X = b_Y = b_Z = 2$.

Build a Diophantine equation as follows:

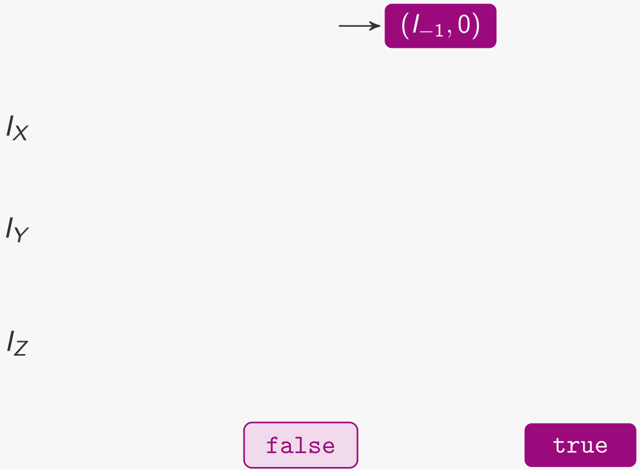
$$0 \cdot l_X - 1 \cdot l_Y + 1 \cdot l_Z = 0$$

Speed up the search

Using MDDs



Speed up the search



Speed up the search

$$0 \cdot l_X - 1 \cdot l_Y + 1 \cdot l_Z = 0 \longrightarrow (l_{-1}, 0)$$

l_X

l_Y

l_Z

`false` `true`

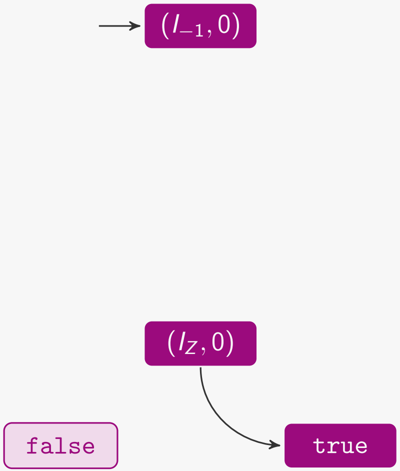
Speed up the search

$$0 \cdot l_X - 1 \cdot l_Y + 1 \cdot l_Z = 0$$

l_X

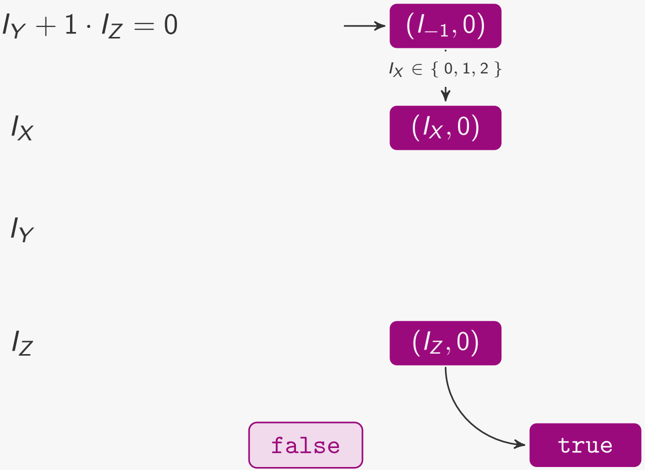
l_Y

l_Z



Speed up the search

$$0 \cdot l_X - 1 \cdot l_Y + 1 \cdot l_Z = 0$$



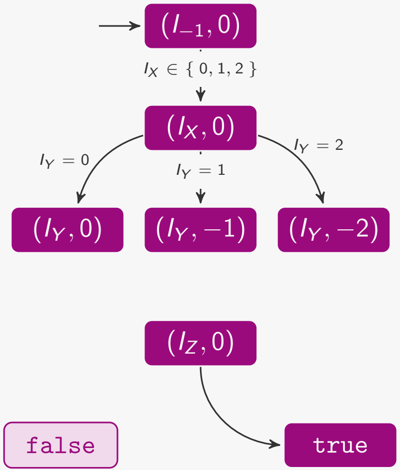
Speed up the search

$0 \cdot l_X - 1 \cdot l_Y + 1 \cdot l_Z = 0$

l_X

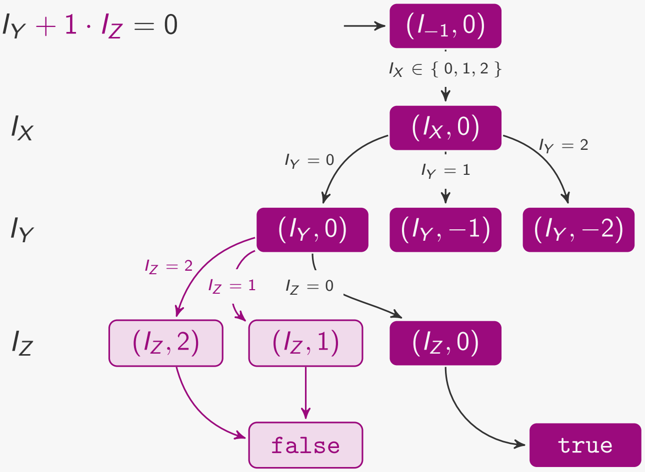
l_Y

l_Z



Speed up the search

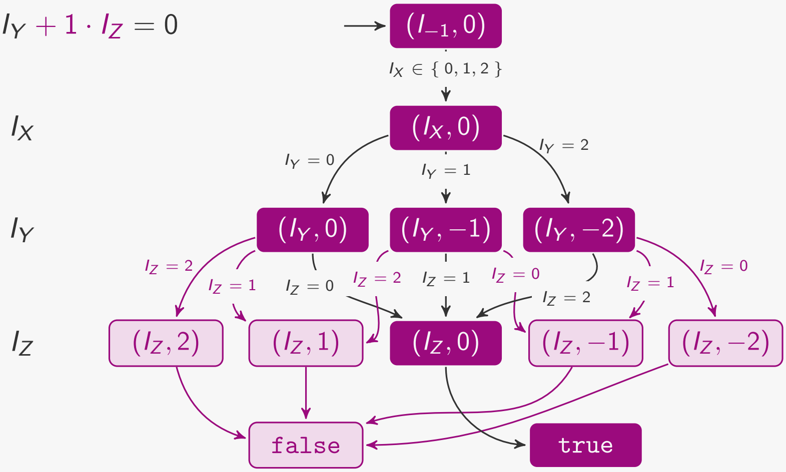
$$0 \cdot I_X - 1 \cdot I_Y + 1 \cdot I_Z = 0$$



Speed up the search

Using MDDs

$$0 \cdot l_X - 1 \cdot l_Y + 1 \cdot l_Z = 0$$



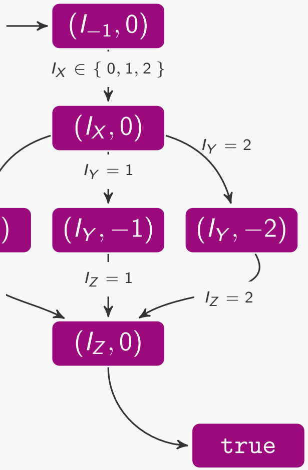
Speed up the search

$0 \cdot l_X - 1 \cdot l_Y + 1 \cdot l_Z = 0$

l_X

l_Y

l_Z



Linear Constraints

How to involve them?

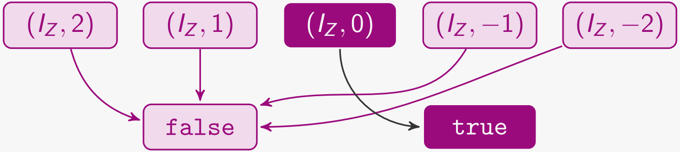
Just connect all corresponding partial sums to the `true` node!

Linear Constraints

How to involve them?

Just connect all corresponding partial sums to the `true` node!

$$0 \cdot I_X - 1 \cdot I_Y + 1 \cdot I_Z \leq 0$$

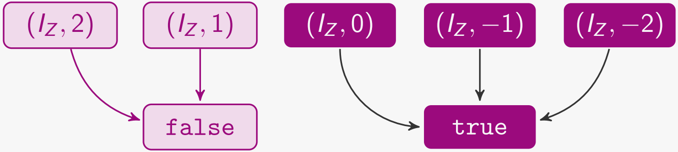


Linear Constraints

How to involve them?

Just connect all corresponding partial sums to the `true` node!

$$0 \cdot l_X - 1 \cdot l_Y + 1 \cdot l_Z \leq 0$$



The Implementation

- Implementation of this pure SAT based approach in a tool called Woorpje.
- Based on the Glucose SAT-Solver
- Competitive and reliable

The Implementation

- Implementation of this pure SAT based approach in a tool called Woorpje.
- Based on the Glucose SAT-Solver
- Competitive and reliable

Woorpje is available at

`http://informatik.uni-kiel.de/~mku/woorpje`

Solving Word Equations Using SAT

Joel D. Day, Thorsten Ehlers, **Mitja Kulczynski**, Florin Manea, Dirk Nowotka and Danny B. Poulsen

RP ’19