Monadic Decomposability of Regular Relations

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Reachability Problems 2019

What is monadic decomposability?

- For a relation, monadic decomposability captures the notion of sufficient independence of the components of the relation.
- Components of $X \times Y$ are completely independent. While in $\{(x, x) \mid x \in X\}$ the components are tightly coupled.

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- Monadic decomposable relation R is expressible as a finite union of direct products of unary predicates.
- It is a powerful tool for decision procedures in logical theories.
- Restricting analysis to monadic decomposable relations can turn an undecidable problem into a decidable one.
- They also provide a large and robust class that can be implemented in the context of SMT solvers.

Main problem

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We focus on relations defined by multitape automata.

Automata with multiple tapes

- Introduced in the 50s.
- Consists of a finite state-control and one-way read-only heads.
- Operates on finite inputs.
- An input is accepted, if a state reached after reading the whole input is final.
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- Represented as a labelled directed graph.

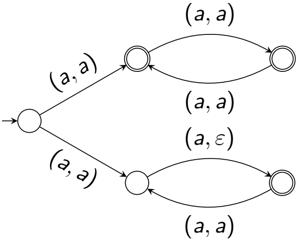
Syntactic restrictions

We study classes based on restrictions on the underlying graph.

Automata with multiple tapes

Variant 1

Nondeterministic behaviour with no restriction on tuples being read.



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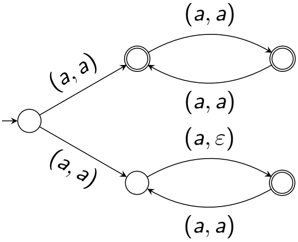
We call such relations *rational* and denote by **Rat**.



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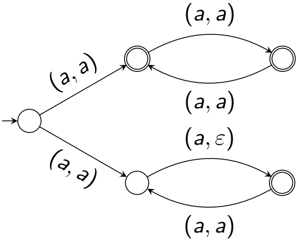

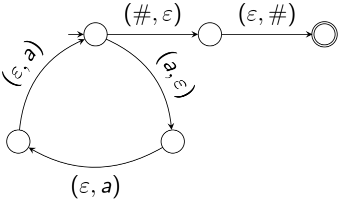

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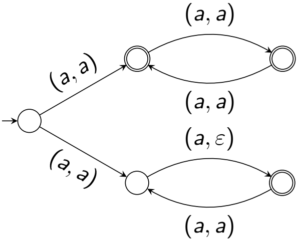
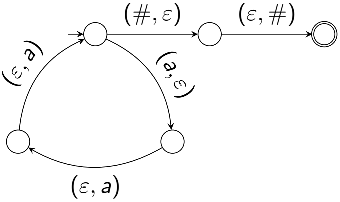
Variant 2

Deterministic behaviour with no restriction on tuples being read.

We call such relations *deterministic rational* and denote by **DRat**.





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Also known as *synchronous* or *automatic* relations.

Regular relation example

Consider $R_{\mathsf{eq}} = \{(u, u) \mid u \in \Sigma^*\}$, where $\Sigma = \{a, b\}$.

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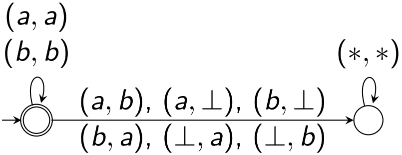

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Let us call the relations $R \subseteq \Sigma^* \times \cdots \times \Sigma^*$ recognized by such automata *recognizable* and denote the class by **Rec**.

Automata with multiple tapes

Rec can be equivalently seen as a finite union of direct products of regular languages.

$$R \in \textbf{Rec} \text{ iff } R = \bigcup_{i=1}^{n} L_{i,1} \times L_{i,2} \times \cdots \times L_{i,k},$$

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Recognizable relations are also known as monadic decomposable relations.

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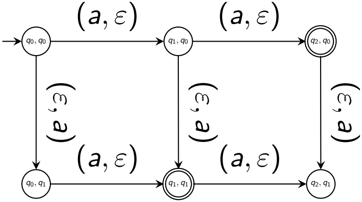

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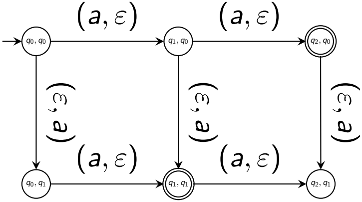

$$R_{\text{fin}} = (\{a\} \times \{a\}) \cup (\{a^2\} \times \{\varepsilon\})$$

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$k = 1$

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For $k > 1$:

Theorem

There exists a strict hierarchy of families:

$$\mathbf{Rec} \subsetneq \mathbf{Reg} \subsetneq \mathbf{DRat} \subsetneq \mathbf{Rat}$$

Relative decidability

Given a relation R in $X \in \{\mathbf{Reg}, \mathbf{DRat}, \mathbf{Rat}\}$. Is R in $Y \in \{\mathbf{Rec}, \mathbf{Reg}, \mathbf{DRat}\}$ as well? (wlog $Y \subsetneq X$)

That is, can R be expressed by a simpler family?

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Theorem (BHLLN'19)

Let R be a regular k-ary relation given by a DFA (resp. NFA). Deciding whether R is monadic decomposable is in NL *(resp.* PSPACE*).*

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Theorem (BHLLN'19)

Let R be a regular k-ary relation given by a DFA (resp. NFA). Deciding whether R is monadic decomposable is in NL *(resp.* PSPACE*). Matching lower bounds also hold.*

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Equivalence relation

Let R be a regular binary relation. Define equivalence relation $\sim$ as

$$u \sim v \text{ iff } \forall w : ((u, w) \in R \Leftrightarrow (v, w) \in R) \land ((w, u) \in R \Leftrightarrow (w, v) \in R)$$

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Lemma (folklore)

Let R be a regular binary relation. Then the equivalence relation $\sim$ has infinite index iff R is not monadic decomposable.

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There are four equivalence classes, $\{\varepsilon\}$, $\{a\}$, $\{a^2\}$ and "everything else". Hence, the relation is monadic decomposable.

Key techniques

- Previous works studied whether a relation R is monadic decomposable.
- We focus on being not decomposable.
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- In a way, we are reasoning on infinite objects rather than finite.
- We will show
  - R is not **Rec** iff there exists a sequence of representatives with a *nice* structure — utilizing combinatorial arguments such as Infinite Ramsey Theorem; and
  - how to generate such a sequence — utilizing pumping argument.

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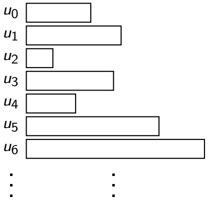
- *and be efficient*.

An infinite sequence of representatives



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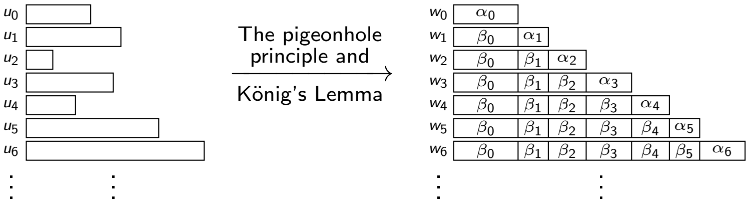

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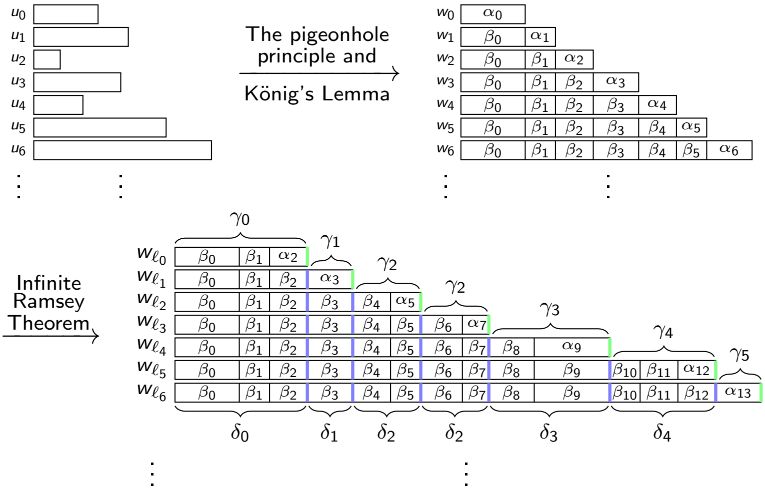

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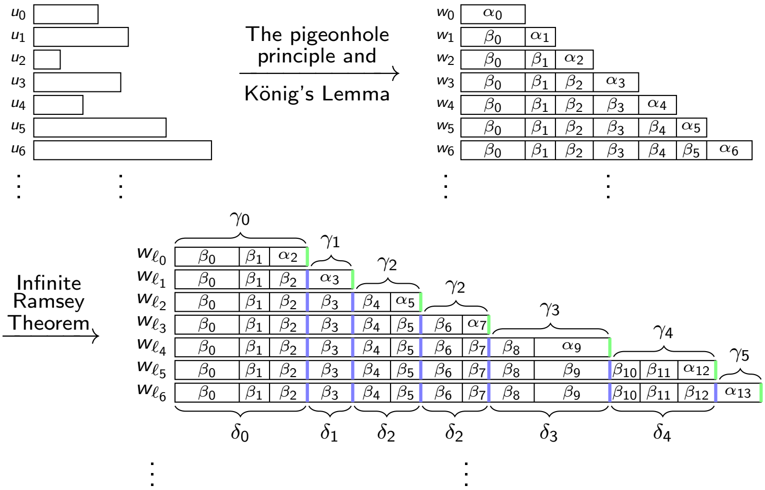

In this sequence, no matter which pair of words is being read, the states are the same after reading (δ_i, δ_i)!

Properties of the sequence

We have an infinite sequence of words $\{w_i\}_{i \geq 0}$, where

- $w_j \not\sim w_\ell$ for all $j \neq \ell$;
- $w_i = \delta_0 \cdots \delta_{i-1} \gamma_i$;
- $|\gamma_i| = |\delta_i| > 0$;
- No matter which pair of words is read, the automaton visits the same states in particular points of computation in $R^{\not\sim}$.

This sequence exists if and only if R is not monadic decomposable.

Synchronizing states and witnesses

We can further prove that R is not monadic decomposable iff there are synchronizing states with pumping property in $R^{\not\sim}$.

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- If R is given by a DFA, then $R^{\not\sim}$ can be constructed in L.
- If R is given by an NFA, then $R^{\not\sim}$ can be constructed in PSPACE.

Main result

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Thank you for your attention!