# On Relevant Equilibria in Reachability Games

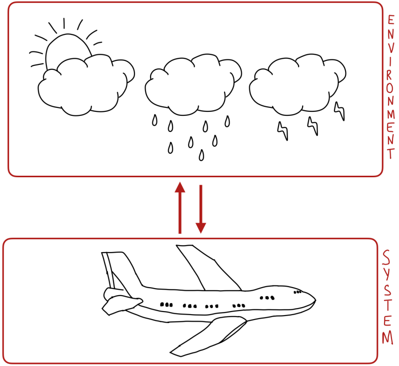
Thomas Brihaye[1] Véronique Bruyère[1] Aline Goeminne[1,2] Nathan Thomasset[1,3]

1. Université de Mons (UMONS), Mons, Belgium.
2. Université libre de Bruxelles (ULB), Brussels, Belgium.
3. ENS Paris-Saclay, Université Paris-Saclay, Cachan, France.

RP'19 – September 11, 2019

# Context

# Verification and synthesis

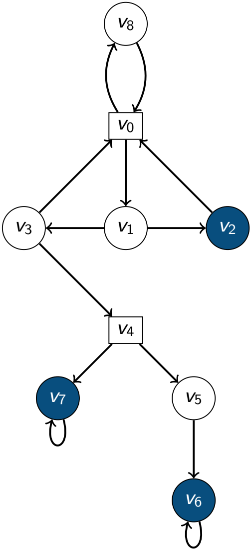



- **Verification:** **checking** that the **system** satisfies some **specifications**.
- **Synthesis:** **building** a system which satisfies some specifications by construction.

  $\hookrightarrow$ games played on graph.

# Two player zero-sum games

# Qualitative two-player zero-sum reachability games



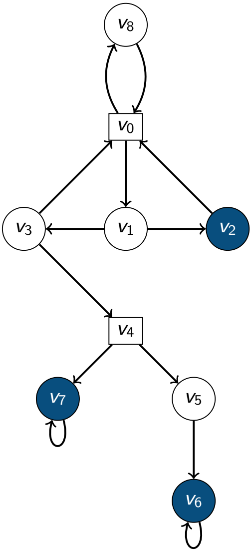

- Player ○: **the system**
  Goal: **satisfying a property.**
  Here: reaching a vertex of the target set $F_{\bigcirc} = \{v_2, v_6, v_7\}$ (**reachability objective**)

- Player □: **the environment**
  Goal: **avoid that.**

> The system satisfies the property
> $\Leftrightarrow$
> Player ○ has a **winning strategy**.

Too restrictive $\rightsquigarrow$ **quantitative** specification.
(Ex: reaching a vertex of the target set within $k$ steps.)

# Quantitative two-player zero-sum reachability games




- **Two** players: Player ◯ (Min) and Player ☐ (Max).

- (**Quantitative** reachability objective) For every infinite path (called **play**) $\rho$, $\rho = \rho_0\rho_1\ldots$,

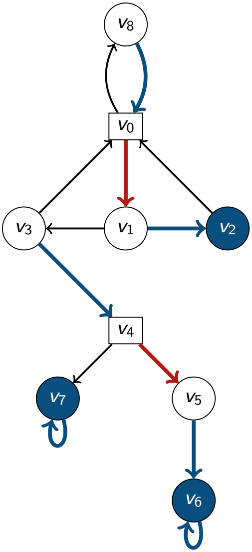
$$\text{Cost}_{\bigcirc}(\rho) = \begin{cases} k & \text{if } k \text{ is the least index st. } \rho_k \in F_{\bigcirc} \\ +\infty & \text{otherwise} \end{cases}$$

Ex:

- $\text{Cost}_{\bigcirc}((v_0v_1v_2)^\omega) = 2$;
- $\text{Cost}_{\bigcirc}((v_0v_8)^\omega) = +\infty$.

- Objectives:
  - Player ◯ wants to reach $F_{\bigcirc}$ ASAP;
  - Player ☐ wants to **avoid** that.

# Quantitative two-player zero-sum reachability games




- Strategy: $\sigma_i : V^* V_i \to V$;
  Ex: $\sigma_{\bigcirc}$ and $\sigma_{\Box}$
- A strategy profile: $(\sigma_{\bigcirc}, \sigma_{\Box}) \rightsquigarrow$
  $\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = (v_0 v_1 v_2)^{\omega}$ (called **outcome**)

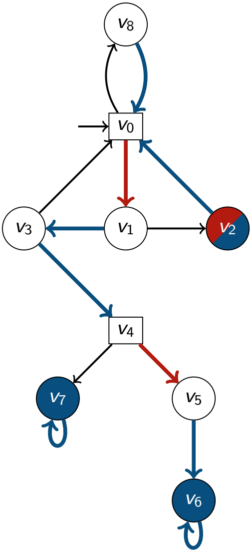
**What cost can Player $\bigcirc$ ensure?**

- From $v_0$, Player $\bigcirc$ can ensure a cost of $+\infty$;
- From $v_3$, Player $\bigcirc$ can ensure a cost of 3;

$\rightsquigarrow$ **value** of a vertex
$\rightsquigarrow$ ~~winning strategy~~ $\rightsquigarrow$ **optimal strategies**.

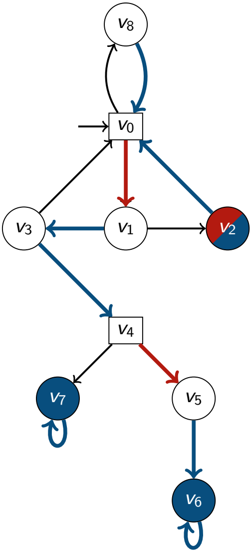
# Multiplayer (non zero-sum) quantitative reachability games

# Setting




- **Two** (or more) players;
  Ex: Player ◯ and Player □.
- Objectives:
  - Player ◯ wants to reach $F_{\bigcirc} = \{v_2, v_6, v_7\}$ (ASAP);
  - Player □ wants to **reach** $F_{\square} = \{v_2\}$ (ASAP).
  - $\rightsquigarrow$ non antagonistic.

# Definition of Nash equilibrium



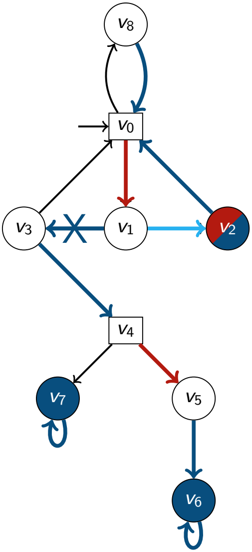

- ~~optimal strategies~~ $\rightsquigarrow$ other solution concept: **Nash equilibrium**.

**Nash equilibrium**

A **strategy profile** $(\sigma_{\bigcirc}, \sigma_{\square})$ is a Nash equilibrium (NE) if **no** player has an **incentive to deviate unilaterally**.

- Counter-ex: $(\sigma_{\bigcirc}, \sigma_{\square})$:
  - $(\sigma_{\bigcirc}, \sigma_{\square}) \rightsquigarrow \langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = v_0 v_1 v_3 v_4 v_5 v_6^{\omega}$;
  - $(\text{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0}), \text{Cost}_{\square}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0})) = (5, +\infty)$.

# Definition of Nash equilibrium




- ~~optimal strategies~~ $\rightsquigarrow$ other solution concept: **Nash equilibrium**.

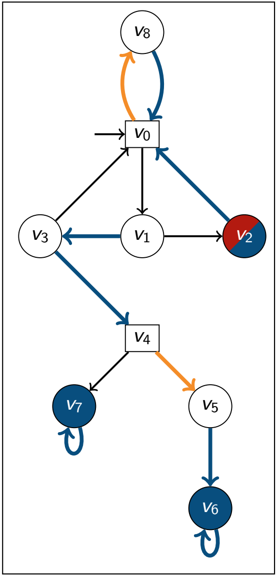
**Nash equilibrium**

A **strategy profile** $(\sigma_{\bigcirc}, \sigma_{\square})$ is a Nash equilibrium (NE) if **no** player has an **incentive to deviate unilaterally**.

- Counter-ex: $(\sigma_{\bigcirc}, \sigma_{\square})$:
  - $(\sigma_{\bigcirc}, \sigma_{\square}) \rightsquigarrow \langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = v_0 v_1 v_3 v_4 v_5 v_6^{\omega}$;
  - $(\mathrm{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0}), \mathrm{Cost}_{\square}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0})) = (5, +\infty)$.

$\rightsquigarrow$ not an NE.

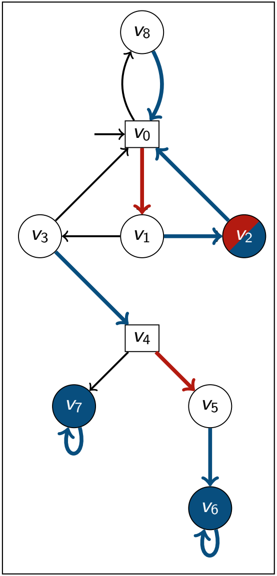
# Different NEs may coexist




- $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = (v_0 v_8)^\omega$
- Cost : $(+\infty, +\infty)$
- **NO player** visits his target set ...

---

- $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = (v_0 v_1 v_2)^\omega$
- Cost : $(2, 2)$
- **BOTH players** visit their target set !



What is (for us) a relevant **Nash equilibrium** ?

# Studied problems

1. **(Threshold decision problem)**

2. **(Social welfare decision problem)**
3. **(Pareto optimal decision problem)**

# Studied problems

1. **(Threshold decision problem)** Given $(k_1, \dots, k_n) \in (\mathbb{N} \cup \{+\infty\})^n$, does there exist an NE $(\sigma_1, \dots, \sigma_n)$ such that, for all $1 \leq i \leq n$:

$$\mathsf{Cost}_i(\langle \sigma_1, \dots, \sigma_n \rangle_{v_0}) \leq k_i.$$

For NEs, in multiplayer quantitative reachability games, Problem 1 is **NP-complete**.

# Key idea

**Outcome characterization of a Nash equilibrium**

Let $\rho$ be a play,
there exists an NE $(\sigma_1, \dots, \sigma_n)$ such that $\langle \sigma_1, \dots, \sigma_n \rangle_{v_0} = \rho$
if and only if
$\rho$ satisfies a "good" property.

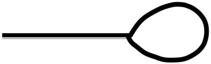
$\rightsquigarrow$ Does there exist a play $\rho$ such that:

- for each player $i$, $\mathrm{Cost}_i(\rho) \leq k_i$;
- $\rho$ satisfies a "good" property?

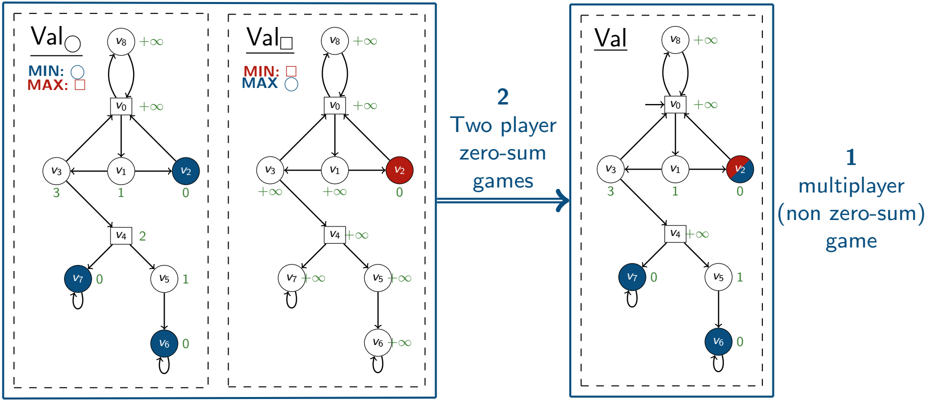
# Algorithm (For NE)

1. it guesses a lasso of polynomial length;
2. it verifies that the cost profile of this lasso satisfies the conditions given by the problem;
3. it verifies that the lasso is the outcome of an NE.

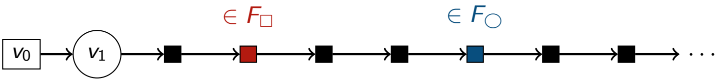
**NP-algorithm for Problem 1:**

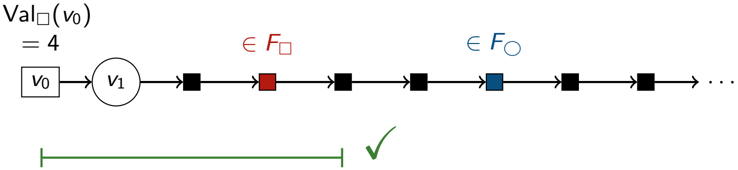
- **Step 1:** if there exists an NE which satisfies the constraints, there exists one which also satisfies the constraints and such that its outcome is a **lasso** $(h\ell^{\omega})$ with a  **polynomial length** $(|h\ell|)$.
- **Step 2:** can be done in **polynomial time.**
- **Step 3:** checking the "good" property along the lasso of polynomial length can be done in **polynomial time.**

# What is this “good” property ?

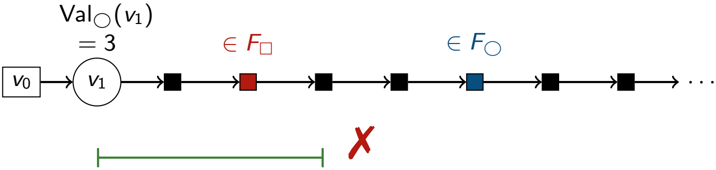


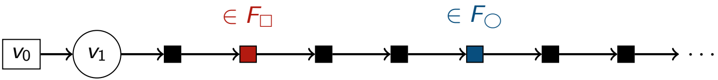

**Values** in quantitative two-player zero-sum games can be computed in **polynomial time** (see for example [BGHM17])


$\in F_{\Box}$
$\in F_{\bigcirc}$
$v_0$
$v_1$
· · ·



$\mathrm{Val}_{\square}(v_0)$
$= 4$
$\in F_{\square}$
$\in F_{\bigcirc}$
$v_0$
$v_1$

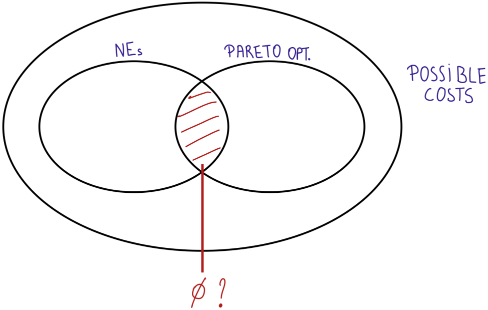

$\mathrm{Val}_{\bigcirc}(v_1)$
$= 3$
$\in F_{\square}$
$\in F_{\bigcirc}$
$v_0$
$v_1$
$\cdots$




- ✓ ✓ ✓ ✓ ✓ . . . ⇝ outcome of an NE;
- ✓ ✓ ✗ ⇝ ~~outcome of an NE~~.

# Conclusion and additional results

1. **(Threshold decision problem)**
2. **(Social welfare decision problem)**
3. **(Pareto optimal decision problem)**



# Results

| Complexity | Qual. Reach. | | Quant. Reach. | |
|---|---|---|---|---|
| | NE | SPE | NE | SPE |
| Prob. 1 | NP-c [CFGR16] | PSPACE-c[BBGR18] | NP-c | PSPACE-c[BBG+19] |
| Prob. 2 | NP-c | PSPACE-c | NP-c | PSPACE-c |
| Prob. 3 | NP-h/$\Sigma_2^P$ | PSPACE-c | NP-h/$\Sigma_2^P$ | PSPACE-c |

| Memory | Qual. Reach. | | Quant. Reach. | |
|---|---|---|---|---|
| | NE | SPE | NE | SPE |
| Prob. 1 | Poly.[CFGR16] | Expo.[BBGR18] | Poly. | Expo. |
| Prob. 2 | Poly. | Expo. | Poly. | Expo. |
| Prob. 3 | Poly. | Expo. | Poly. | Expo. |

# References


Thomas Brihaye, Véronique Bruyère, Aline Goeminne, Jean-François Raskin, and Marie van den Bogaard, The complexity of subgame perfect equilibria in quantitative reachability games, 30th International Conference on Concurrency Theory, CONCUR 2019, August 27-30, 2019, Amsterdam, the Netherlands., 2019, pp. 13:1–13:16.

Thomas Brihaye, Véronique Bruyère, Aline Goeminne, and Jean-François Raskin, Constrained existence problem for weak subgame perfect equilibria with $\omega$-regular boolean objectives, Proceedings Ninth International Symposium on Games, Automata, Logics, and Formal Verification, GandALF 2018, Saarbrücken, Germany, 26-28th September 2018., 2018, pp. 16–29.

Thomas Brihaye, Gilles Geeraerts, Axel Haddad, and Benjamin Monmege, Pseudopolynomial iterative algorithm to solve total-payoff games and min-cost reachability games, Acta Inf. **54** (2017), no. 1, 85–125.

Rodica Condurache, Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin, The Complexity of Rational Synthesis, 43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016) (Dagstuhl, Germany) (Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi, eds.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 55, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016, pp. 121:1–121:15.