

Properties of the algebraic theories all of whose algebras are free

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Joint work with Tom Leinster

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Slides: <https://webhomes.maths.ed.ac.uk/~mwoolf/>

Each algebraic theory (finitary monad on **Set**; Lawvere theory) has some free algebras.

Which theories have *all* their algebras being free?

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sets with at most one element

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- ▶ 1970s, logic: Steven Givant
- ▶ 2010s, classical universal algebra: Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei

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Properties: affine vs. pointed, minimality, dimension

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Example: k -vector spaces $(\alpha_i, \alpha'_i \in k)$.

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Set $x_2 = \cdots = x_m = 0$. Then

$$\beta_1 x_1 = 0, \quad \text{i.e.} \quad x_1 = 0.$$

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$$\begin{array}{ccc} T & & T\text{-Alg} \\ \downarrow & & \uparrow \\ S & & S\text{-Alg} \end{array}$$

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Assume all T -algebras are free. Assume we've not collapsed the theory down to a degenerate one. Then minimality says “ $S\text{-Alg}$ must be all of $T\text{-Alg}$ ”.

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Fact: for large enough N , we have $TX \hookrightarrow TQ \cong P^N$ in $T\text{-Alg}$ (via $X \hookrightarrow Q$).

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Fact: for large enough N , we have $TX \hookrightarrow TQ \cong P^N$ in $T\text{-Alg}$ (via $X \hookrightarrow Q$).

So TX is a subalgebra of an S -algebra, hence $TX \in S\text{-Alg}$.

Dimension

Dimension: “If $TX \cong TY$, then $X \cong Y$.”

That is, the free functor is essentially *injective* on objects.

All algebras being free is the free functor being essentially surjective on objects.

So essential surjectivity of the free functor $\implies$ essential injectivity.

Dimension

Lemma 1. *Let T be a finitary monad on **Set**.*

Let $D : \mathbf{I} \rightarrow \mathbf{Set}^T$ be a filtered diagram such that the algebra $D(i)$ is finitely generated for each object i of $\mathbf{I}$, and the homomorphism $D(u) : D(i) \rightarrow D(j)$ is surjective for each map $u : i \rightarrow j$ in $\mathbf{I}$.

Let $n \in \mathbb{N}$. If $\text{colim } D$ can be generated by n elements, then so can $D(i)$ for some $i \in \mathbf{I}$.

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Does anyone know of a reference for this please?

Please send to maia.woolf@ed.ac.uk. Thanks in advance!

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Lemma 2. *Let T be a totally free theory, and let $n \in \mathbb{N}$. Then every surjective homomorphism $Tn \twoheadrightarrow Tn$ is an isomorphism.*

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Proof sketch. Consider the preordered set with elements the surjective homomorphisms $s : Tn \twoheadrightarrow Tn$, and $s \leq s'$ iff there is a homomorphism t such that the following commutes:

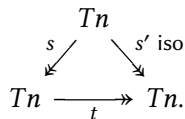
$$\begin{array}{ccc} & Tn & \\ s \swarrow & & \searrow s' \\ Tn & \xrightarrow{\quad t \quad} & Tn. \end{array}$$

Goal: show that every element is an iso.

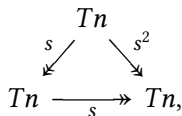
Dimension

Key facts:

- Isomorphisms are downwards closed:



- $s \leq s^2$ always:



so s maximal $\implies s$ an iso.

One can show that every chain has an upper bound.

Then Zorn's Lemma says each element is less than a maximal element!

So every element is an iso.

Thank you!

Slides & bibliography:

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