

LOCAL PRESENTABILITY OF (CO)MONADS IN DOUBLE CATEGORIES

Theofilos Tsantilas

joint work with V. Aravantinos-Sotiropoulos and C. Vasilakopoulou

School of Applied Mathematical and Physical Sciences, National Technical University of Athens

PSSL113 - UCLouvain
September 26, 2027

This work was implemented in the framework of H.F.R.I call “3rd Call for H.F.R.I.’s Research Projects to Support Faculty Members & Researchers” (H.F.R.I. Project Number: 23249)



SECTION 0

Motivation

■ Proposition [Porst]

If $\mathcal{V}$ is a *locally presentable*, symmetric monoidal category whose *tensor product is finitary in each variable* then

- 1 $\text{Mon}(\mathcal{V})$ is finitary monadic over $\mathcal{V}$ and locally presentable;
- 2 $\text{Comon}(\mathcal{V})$ is comonadic over $\mathcal{V}$ and locally presentable.

■ Proposition [Porst]

If $\mathcal{V}$ is a *locally presentable*, symmetric monoidal category whose *tensor product is finitary in each variable* then

- 1 $\text{Mon}(\mathcal{V})$ is finitary monadic over $\mathcal{V}$ and locally presentable;
- 2 $\text{Comon}(\mathcal{V})$ is comonadic over $\mathcal{V}$ and locally presentable.

Question: Under what conditions can we generalize these results to the context of double categories?

$$\begin{array}{ccc} \text{monoidal } \mathcal{V} & \equiv & \begin{array}{c} \text{1-object} \\ \text{bicategory} \end{array} \xrightarrow{\text{generalize to}} \text{double category } \mathbb{D} \\ \\ \text{(co)monoids in } \mathcal{V} & \xrightarrow{\text{pass to}} & \text{(co)monads in } \mathbb{D} \end{array}$$

SECTION 1

Locally presentable double categories

Many examples where the same objects can have two kinds of arrows which usually coexist and interact. Double categories provide a 2-dimensional categorical framework for such contexts.

■ Definition

A *(pseudo) double category* is a (pseudo) category object inside $\mathbf{Cat}$.

Many examples where the same objects can have two kinds of arrows which usually coexist and interact. Double categories provide a 2-dimensional categorical framework for such contexts.

Definition

A **(pseudo) double category** is a (pseudo) category object inside $\mathbf{Cat}$.

So, a double category $\mathbb{D}$ consists of a **category** $\mathbb{D}_0$ of objects and a **category** $\mathbb{D}_1$ of arrows, and source, target, composition and identity **functors**

$$\begin{array}{ccccc} & \xleftarrow{s} & & & \\ \mathbb{D}_0 & \xrightarrow{1} & \mathbb{D}_1 & \xleftarrow{\odot} & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \\ & \xleftarrow{t} & & & \end{array}$$

Subject to the usual axioms and *coherence conditions*.

Many examples where the same objects can have two kinds of arrows which usually coexist and interact. Double categories provide a 2-dimensional categorical framework for such contexts.

Definition

A *(pseudo) double category* is a (pseudo) category object inside $\mathbf{Cat}$.

So, a double category $\mathbb{D}$ consists of a *category* $\mathbb{D}_0$ of objects and a *category* $\mathbb{D}_1$ of arrows, and source, target, composition and identity *functors*

$$\begin{array}{c} \xleftarrow{s} \\ \mathbb{D}_0 \xrightarrow{1} \mathbb{D}_1 \xleftarrow{\odot} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \\ \xleftarrow{t} \end{array}$$

Subject to the usual axioms and *coherence conditions*.

Takeaway: A double category $\mathbb{D}$ consists of:

- ⊙ *objects* $X, Y, Z, \dots$;
- ⊙ *tight arrows* which we denote as $X \rightarrow Y$;
- ⊙ *loose arrows* which we denote as $X \rightharpoonup Y$; and
- ⊙ *cells* of the form

$$\begin{array}{ccc} X & \xrightarrow{P} & Y \\ f \downarrow & \alpha & \downarrow g \\ Z & \xrightarrow{Q} & W \end{array}$$

In many important examples the tight and the loose arrows are more closely related.

■ Definition [Shulman]

A pseudo double category $\mathbb{D}$ is called **fibrant** if $\langle s, t \rangle : \mathbb{D}_1 \rightarrow \mathbb{D}_0 \times \mathbb{D}_0$ is a (bi)fibration.

In a fibrant double category, we can *turn tight arrows to loose arrows* thus obtaining a rich *interplay between the categories $\mathbb{D}_0$ and $\mathbb{D}_1$* .

In many important examples the tight and the loose arrows are more closely related.

Definition [Shulman]

A pseudo double category $\mathbb{D}$ is called **fibrant** if $\langle s, t \rangle : \mathbb{D}_1 \rightarrow \mathbb{D}_0 \times \mathbb{D}_0$ is a (bi)fibration.

In a fibrant double category, we can *turn tight arrows to loose arrows* thus obtaining a rich *interplay between the categories $\mathbb{D}_0$ and $\mathbb{D}_1$* .

Takeaway: For any tight arrow $f : X \rightarrow Y$ we get two loose arrows, its **companion** $\hat{f} : X \rightarrowtail Y$ and its **conjoint** $\check{f} : Y \rightarrowtail X$ together with cells

$$\begin{array}{ccccc}
 X & \xrightarrow{\hat{f}} & Y & X & \xlongequal{\quad} & X & Y & \xrightarrow{\check{f}} & X & X & \xlongequal{\quad} & X \\
 f \downarrow & p_1 & \parallel & \parallel & p_2 & \downarrow f & \parallel & q_1 & \downarrow f & f \downarrow & q_2 & \parallel \\
 Y & \xlongequal{\quad} & Y & X & \xrightarrow{\hat{f}} & Y & Y & \xlongequal{\quad} & Y & Y & \xrightarrow{\check{f}} & X
 \end{array}$$

subject to certain conditions.

Example: the double category of (co)spans

Suppose $\mathcal{C}$ is a category with pullbacks.

The double category $\text{Span}_{\mathcal{C}}$ of spans in $\mathcal{C}$ consists of *objects*, *arrows*, *spans*, and *span homomorphisms in $\mathcal{C}$* . A typical cell here is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{P} & Z \\ f \downarrow & \alpha & \downarrow g \\ X' & \xrightarrow{Q} & Z' \end{array} \equiv \begin{array}{ccccc} X & \longleftarrow & Y & \longrightarrow & Z \\ f \downarrow & & \downarrow \alpha & & \downarrow g \\ X' & \longleftarrow & Y' & \longrightarrow & Z' \end{array} \quad \text{in } \mathcal{C}.$$

Composition of spans is given by *pullbacks*, the identities are the evident ones.

Example: the double category of (co)spans

Suppose $\mathcal{C}$ is a category with pullbacks.

The double category $\text{Span}_{\mathcal{C}}$ of spans in $\mathcal{C}$ consists of *objects*, *arrows*, *spans*, and *span homomorphisms in $\mathcal{C}$* . A typical cell here is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{P} & Z \\ f \downarrow & \alpha & \downarrow g \\ X' & \xrightarrow{Q} & Z' \end{array} \equiv \begin{array}{ccccc} X & \longleftarrow & Y & \longrightarrow & Z \\ f \downarrow & & \downarrow \alpha & & \downarrow g \\ X' & \longleftarrow & Y' & \longrightarrow & Z' \end{array} \quad \text{in } \mathcal{C}.$$

Composition of spans is given by *pullbacks*, the identities are the evident ones.

$\text{Span}_{\mathcal{C}}$ is *fibrant*: any arrow $f : X \rightarrow Y$ induces the *companion* span

$$X \rightrightarrows X \xrightarrow{f} Y$$

and the *conjoint* span

$$Y \xleftarrow{f} X \rightrightarrows X.$$

Dually, we can form the double category $\text{coSpan}_{\mathcal{C}}$ of cospans in $\mathcal{C}$.

Example: the double category of relations

If $\mathcal{C}$ is also *regular*, we can form the double category $\mathbb{R}el_{\mathcal{C}}$ which consists of *objects*, *arrows*, *relations*, and *inclusions of relations in $\mathcal{C}$* :

$$\begin{array}{ccc} X & \xrightarrow{P} & Y \\ f \downarrow & \alpha & \downarrow g \\ Z & \xrightarrow{Q} & W \end{array} \iff gPf^{\circ} \subseteq Q.$$

Compositions and identities are the evident ones.

Example: the double category of relations

If $\mathcal{C}$ is also *regular*, we can form the double category $\mathbb{R}el_{\mathcal{C}}$ which consists of *objects*, *arrows*, *relations*, and *inclusions of relations in $\mathcal{C}$* :

$$\begin{array}{ccc} X & \xrightarrow{P} & Y \\ f \downarrow & \alpha & \downarrow g \\ Z & \xrightarrow{Q} & W \end{array} \iff gPf^{\circ} \subseteq Q.$$

Compositions and identities are the evident ones.

$\mathbb{R}el_{\mathcal{C}}$ is *fibrant*: any arrow $f : X \rightarrow Y$ induces two relations, its *graph* (companion) which is f viewed as a relation, and its *opposite graph* (conjoint) f° .

Example: $\mathcal{V}$ -matrices

Suppose $\mathcal{V}$ is a *monoidal category with small coproducts preserved by $\otimes$ in each variable*.

The double category $\mathcal{V}\text{-Mat}$ consists of *sets, functions, families $\{P_{y,x}\}_{(y,x) \in Y \times X}$ of objects of $\mathcal{V}$ with X, Y sets called $\mathcal{V}$ -matrices and morphisms between families*. Which means that a cell is a family of functions

$$\begin{array}{ccc}
 X & \xrightarrow{P} & Y \\
 f \downarrow & \alpha & \downarrow g \\
 Z & \xrightarrow{Q} & W
 \end{array}
 \iff
 \{\alpha_{y,x} : P_{y,x} \rightarrow Q_{g(y),f(x)}\}_{(y,x) \in Y \times X}$$

Composition of two $\mathcal{V}$ -matrices $P : X \rightarrowtail Y$ and $Q : Y \rightarrowtail Z$ is given by the family $\left\{ \sum_{y \in Y} Q_{z,y} \otimes P_{y,x} \right\}$.

Example: $\mathcal{V}$ -matrices

Suppose $\mathcal{V}$ is a *monoidal category with small coproducts preserved by $\otimes$ in each variable*.

The double category $\mathcal{V}\text{-Mat}$ consists of *sets, functions, families $\{P_{y,x}\}_{(y,x) \in Y \times X}$ of objects of $\mathcal{V}$ with X, Y sets called $\mathcal{V}$ -matrices and morphisms between families*. Which means that a cell is a family of functions

$$\begin{array}{ccc}
 X & \xrightarrow{P} & Y \\
 f \downarrow & \alpha & \downarrow g \\
 Z & \xrightarrow{Q} & W
 \end{array}
 \iff
 \{\alpha_{y,x} : P_{y,x} \rightarrow Q_{g(y),f(x)}\}_{(y,x) \in Y \times X}$$

Composition of two $\mathcal{V}$ -matrices $P : X \rightarrowtail Y$ and $Q : Y \rightarrowtail Z$ is given by the family $\{\sum_{y \in Y} Q_{z,y} \otimes P_{y,x}\}$.

$\mathcal{V}\text{-Mat}$ is *fibrant*: for any function $f : X \rightarrow Y$, its companion and conjoint are given by the same formula,

$$\{I \text{ if } f(x) = y \text{ and } 0 \text{ otherwise.}\}$$

Definition [Aravantinos-Sotiropoulos, Vasilakopoulou]

A *fibrant* double category $\mathbb{D}$ is called **locally presentable** if:

- ⊙ $\mathbb{D}_0, \mathbb{D}_1$ are *locally presentable*,
- ⊙ s, t are *cocontinuous right adjoints*, and
- ⊙ for every object A of $\mathbb{D}_1$, *the functors $- \odot A$ and $A \odot -$ are accessible*.

Definition [Aravantinos-Sotiropoulos, Vasilakopoulou]

A *fibrant* double category $\mathbb{D}$ is called **locally presentable** if:

- ⊙ $\mathbb{D}_0, \mathbb{D}_1$ are *locally presentable*,
- ⊙ s, t are *cocontinuous right adjoints*, and
- ⊙ for every object A of $\mathbb{D}_1$, *the functors $- \odot A$ and $A \odot -$ are accessible*.

Examples [Aravantinos-Sotiropoulos, Vasilakopoulou]

- 1 $\text{Span}_{\mathcal{C}}$ is locally presentable as long as $\mathcal{C}$ is.
- 2 Rel_{Set} is locally presentable.
- 3 $\mathcal{V}\text{-Mat}$ is locally presentable if $\mathcal{V}$ is and $\otimes$ is accessible in each variable.

SECTION 2

(co)Monads in double categories

A **monad** in a double category $\mathbb{D}$ is some $A_X : X \rightarrow X$ equipped with *multiplication* and *unit* cells

$$\begin{array}{ccc}
 X & \xrightarrow{A} & X & \xrightarrow{A} & X & & X & \xrightarrow{1_X} & X \\
 \parallel & & & & \parallel & & \parallel & & \parallel \\
 & & \mu_A & & & & \eta_A & & \\
 X & \xrightarrow{A} & X & & X & \xrightarrow{A} & X
 \end{array}$$

satisfying the usual associativity and unit laws.

A **monad** in a double category $\mathbb{D}$ is some $A_X : X \rightarrow X$ equipped with *multiplication* and *unit* cells

$$\begin{array}{ccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ \parallel & & \mu_A & & \parallel \\ X & \xrightarrow{A} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \parallel & & \eta_A & & \parallel \\ X & \xrightarrow{A} & X \end{array}$$

satisfying the usual associativity and unit laws. Given two monads A_X, B_Y , a **monad morphism** is a cell $\alpha : A_X \Rightarrow B_Y$ that *respects the multiplication and the units*:

$$\begin{array}{ccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ f \downarrow & \alpha & f \downarrow & \alpha & \downarrow f \\ Y & \xrightarrow{B} & Y & \xrightarrow{B} & Y \\ \parallel & & \mu_B & & \parallel \\ Y & \xrightarrow{B} & Y \end{array} = \begin{array}{ccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ \parallel & & \mu_A & & \parallel \\ X & \xrightarrow{A} & X \\ f \downarrow & \alpha & \downarrow f \\ Y & \xrightarrow{B} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \parallel & & \eta_A & & \parallel \\ X & \xrightarrow{A} & X \\ f \downarrow & \alpha & \downarrow f \\ Y & \xrightarrow{B} & Y \end{array} = \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ f \downarrow & 1_f & \downarrow f \\ Y & \xrightarrow{1_Y} & Y \\ \parallel & & \eta_B & & \parallel \\ Y & \xrightarrow{B} & Y \end{array}$$

A **monad** in a double category $\mathbb{D}$ is some $A_X : X \rightarrow X$ equipped with *multiplication* and *unit* cells

$$\begin{array}{ccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ \parallel & & \mu_A & & \parallel \\ X & \xrightarrow{A} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \parallel & & \eta_A & & \parallel \\ X & \xrightarrow{A} & X \end{array}$$

satisfying the usual associativity and unit laws. Given two monads A_X, B_Y , a **monad morphism** is a cell $\alpha : A_X \Rightarrow B_Y$ that *respects the multiplication and the units*:

$$\begin{array}{ccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ f \downarrow & \alpha & f \downarrow & \alpha & \downarrow f \\ Y & \xrightarrow{B} & Y & \xrightarrow{B} & Y \\ \parallel & & \mu_B & & \parallel \\ Y & \xrightarrow{B} & Y \end{array} = \begin{array}{ccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ \parallel & & \mu_A & & \parallel \\ X & \xrightarrow{A} & X \\ f \downarrow & \alpha & \downarrow f \\ Y & \xrightarrow{B} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \parallel & & \eta_A & & \parallel \\ X & \xrightarrow{A} & X \\ f \downarrow & \alpha & \downarrow f \\ Y & \xrightarrow{B} & Y \end{array} = \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ f \downarrow & 1_f & \downarrow f \\ Y & \xrightarrow{1_Y} & Y \\ \parallel & & \eta_B & & \parallel \\ Y & \xrightarrow{B} & Y \end{array}$$

Monads and monad morphisms assemble into a category, $\mathbf{Mnd}(\mathbb{D})$. Dually, we get **comonads** and **comonad morphisms** which assemble into the category $\mathbf{Cmd}(\mathbb{D})$.

Examples

1 $\mathbf{Mnd}(\mathbf{Span}_{\mathcal{C}}) = \mathbf{Cat}_{\mathcal{C}}$:

loose endo-arrow $A_{X_0} : X_0 \rightrightarrows X_0$	gives	source and target $X_0 \xleftarrow{s} X_1 \xrightarrow{t} X_0$
multiplication $\mu_A : A \odot A \Rightarrow A$		composition $X_1 \times_{X_0} X_1 \rightarrow X_1$
unit $\eta_A : 1_A \Rightarrow A$		identity $X_0 \rightarrow X_1$
monad morphism		internal functor

Comonads in $\mathbf{Span}_{\mathcal{C}}$ are just arrows; the counit implies equality of the legs.

Examples

1 $\text{Mnd}(\text{Span}_{\mathcal{C}}) = \text{Cat}_{\mathcal{C}}$:

loose endo-arrow $A_{X_0} : X_0 \rightharpoonup X_0$	gives	source and target $X_0 \xleftarrow{s} X_1 \xrightarrow{t} X_0$
multiplication $\mu_A : A \odot A \Rightarrow A$		composition $X_1 \times_{X_0} X_1 \rightarrow X_1$
unit $\eta_A : 1_A \Rightarrow A$		identity $X_0 \rightarrow X_1$
monad morphism		internal functor

Comonads in $\text{Span}_{\mathcal{C}}$ are just arrows; the counit implies equality of the legs.

2 $\text{Mnd}(\text{Rel}_{\mathcal{C}}) = \text{Preord}_{\mathcal{C}}$:

loose endo-arrow $R_X : X \rightharpoonup X$	gives	relation $R \subseteq X \times X$
multiplication $\mu_R : R \odot R \Rightarrow R$		$RR \subseteq R \Leftrightarrow R$ is transitive
unit $\eta_R : 1_R \Rightarrow R$		$\Delta_X \subseteq R \Leftrightarrow R$ is reflexive
monad morphism		order preserving morphism

A comonad $C_X : X \rightarrow X \in \text{Cmd}(\text{Rel}_{\mathcal{C}})$ is just a subobject of X ; the counit implies $C \subseteq \Delta_X$.

Examples

1 $\text{Mnd}(\text{Span}_{\mathcal{C}}) = \text{Cat}_{\mathcal{C}}$:

loose endo-arrow $A_{X_0} : X_0 \rightharpoonup X_0$	gives	source and target $X_0 \xleftarrow{s} X_1 \xrightarrow{t} X_0$
multiplication $\mu_A : A \odot A \Rightarrow A$		composition $X_1 \times_{X_0} X_1 \rightarrow X_1$
unit $\eta_A : 1_A \Rightarrow A$		identity $X_0 \rightarrow X_1$
monad morphism		internal functor

Comonads in $\text{Span}_{\mathcal{C}}$ are just arrows; the counit implies equality of the legs.

2 $\text{Mnd}(\text{Rel}_{\mathcal{C}}) = \text{Preord}_{\mathcal{C}}$:

loose endo-arrow $R_X : X \rightharpoonup X$	gives	relation $R \subseteq X \times X$
multiplication $\mu_R : R \odot R \Rightarrow R$		$RR \subseteq R \Leftrightarrow R$ is transitive
unit $\eta_R : 1_R \Rightarrow R$		$\Delta_X \subseteq R \Leftrightarrow R$ is reflexive
monad morphism		order preserving morphism

A comonad $C_X : X \rightarrow X \in \text{Cmd}(\text{Rel}_{\mathcal{C}})$ is just a subobject of X ; the counit implies $C \subseteq \Delta_X$.

3 $\text{Mnd}(\mathcal{V}\text{-Mat}) = \mathcal{V}\text{-Cat}$:

loose endo-arrow $A_X : X \rightharpoonup X$	gives	family of objects $A_{x,x}$ in $\mathcal{V}$
multiplication $\mu_A : A \odot A \Rightarrow A$		morphisms $A_{x',x''} \otimes A_{x,x'} \rightarrow A_{x,x''}$
unit $\eta_A : 1_A \Rightarrow A$		$I \rightarrow A_{x,x'}$
monad morphism		$\mathcal{V}$ -functors

Dually, $\text{Cmd}(\mathcal{V}\text{-Mat}) = \mathcal{V}\text{-coCat}$.

SECTION 3

Local presentability of $\mathbf{Mnd}(\mathbb{D})$ and $\mathbf{Cmd}(\mathbb{D})$

There are obvious forgetful functors

$$\begin{aligned}\mathrm{Mnd}(\mathbb{D}) &\rightarrow \mathrm{End}(\mathbb{D}) \\ (X \overset{A_X}{\rightharpoonup} X, \mu_A, \eta_A) &\mapsto X \overset{A_X}{\rightharpoonup} X\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cmd}(\mathbb{D}) &\rightarrow \mathrm{End}(\mathbb{D}) \\ (Y \overset{C_Y}{\rightharpoonup} Y, \delta_C, \varepsilon_C) &\mapsto Y \overset{C_Y}{\rightharpoonup} Y\end{aligned}$$

There are obvious forgetful functors

$$\begin{aligned} \mathbf{Mnd}(\mathbb{D}) &\rightarrow \mathbf{End}(\mathbb{D}) \\ (X \xrightarrow{A_X} X, \mu_A, \eta_A) &\mapsto X \xrightarrow{A_X} X \end{aligned}$$

and

$$\begin{aligned} \mathbf{Cmd}(\mathbb{D}) &\rightarrow \mathbf{End}(\mathbb{D}) \\ (Y \xrightarrow{C_Y} Y, \delta_C, \varepsilon_C) &\mapsto Y \xrightarrow{C_Y} Y \end{aligned}$$

Then, we proved that *under certain assumptions on $\mathbb{D}$,*

- ⊙ there is a (co)free (co)monad,
- ⊙ (co)monads are (co)monadic over $\mathbf{End}(\mathbb{D})$ as soon as the free (co)monad exists, and
- ⊙ *the categories $\mathbf{Mnd}(\mathbb{D})$ and $\mathbf{Cmd}(\mathbb{D})$ are locally presentable.*

■ Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

If $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits then $\text{Mnd}(\mathbb{D})$ is locally presentable.

Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

If $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits then $\mathbf{Mnd}(\mathbb{D})$ is locally presentable.

Proof. Consider the F -algebras for the endofunctor

$$F : \mathbf{End}(\mathbb{D}) \rightarrow \mathbf{End}(\mathbb{D})$$

$$A_X \mapsto (A_X \odot A_X) + 1_X,$$

that is some $A_X : X \rightarrow X$ in $\mathbf{End}(\mathbb{D})$ equipped with a structure map

$$\alpha : FA_X \Rightarrow A_X \iff \alpha : (A_X \odot A_X) + 1_X \Rightarrow A_X \iff \begin{cases} \alpha_1 : A_X \odot A_X \Rightarrow A_X \\ \alpha_2 : 1_X \Rightarrow A_X \end{cases}$$

Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

If $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits then $\mathbf{Mnd}(\mathbb{D})$ is locally presentable.

Proof. Consider the F -algebras for the endofunctor

$$F : \mathbf{End}(\mathbb{D}) \rightarrow \mathbf{End}(\mathbb{D})$$

$$A_X \mapsto (A_X \odot A_X) + 1_X,$$

that is some $A_X : X \rightarrow X$ in $\mathbf{End}(\mathbb{D})$ equipped with a structure map

$$\alpha : FA_X \Rightarrow A_X \iff \alpha : (A_X \odot A_X) + 1_X \Rightarrow A_X \iff \begin{cases} \alpha_1 : A_X \odot A_X \Rightarrow A_X \\ \alpha_2 : 1_X \Rightarrow A_X \end{cases}$$

F is finitary
 $\mathbf{End}(\mathbb{D})$ is locally presentable since $\mathbb{D}$ is $\left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \mathbf{Alg}(F)$ is locally presentable.

Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

If $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits then $\mathbf{Mnd}(\mathbb{D})$ is locally presentable.

Proof. Consider the F -algebras for the endofunctor

$$F : \mathbf{End}(\mathbb{D}) \rightarrow \mathbf{End}(\mathbb{D})$$

$$A_X \mapsto (A_X \odot A_X) + 1_X,$$

that is some $A_X : X \rightarrow X$ in $\mathbf{End}(\mathbb{D})$ equipped with a structure map

$$\alpha : FA_X \Rightarrow A_X \iff \alpha : (A_X \odot A_X) + 1_X \Rightarrow A_X \iff \begin{cases} \alpha_1 : A_X \odot A_X \Rightarrow A_X \\ \alpha_2 : 1_X \Rightarrow A_X \end{cases}$$

F is finitary
 $\mathbf{End}(\mathbb{D})$ is locally presentable since $\mathbb{D}$ is $\left. \vphantom{\begin{matrix} \alpha : FA_X \Rightarrow A_X \iff \alpha : (A_X \odot A_X) + 1_X \Rightarrow A_X \iff \end{matrix}} \right\} \Rightarrow \mathbf{Alg}(F)$ is locally presentable.

$\mathbf{Mnd}(\mathbb{D})$ is a full subcategory of $\mathbf{Alg}(F)$ closed under limits and filtered colimits. Thus, by the Reflection Theorem, $\mathbf{Mnd}(\mathbb{D})$ is also locally presentable. $\square$

Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

If $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits then $\mathbf{Mnd}(\mathbb{D})$ is locally presentable.

Proof. Consider the F -algebras for the endofunctor

$$F: \mathbf{End}(\mathbb{D}) \rightarrow \mathbf{End}(\mathbb{D})$$

$$A_X \mapsto (A_X \odot A_X) + 1_X,$$

that is some $A_X : X \rightarrow X$ in $\mathbf{End}(\mathbb{D})$ equipped with a structure map

$$\alpha : FA_X \Rightarrow A_X \iff \alpha : (A_X \odot A_X) + 1_X \Rightarrow A_X \iff \begin{cases} \alpha_1 : A_X \odot A_X \Rightarrow A_X \\ \alpha_2 : 1_X \Rightarrow A_X \end{cases}$$

F is finitary
 $\mathbf{End}(\mathbb{D})$ is locally presentable since $\mathbb{D}$ is $\left. \vphantom{\begin{matrix} \alpha : FA_X \Rightarrow A_X \\ \alpha : (A_X \odot A_X) + 1_X \Rightarrow A_X \end{matrix}} \right\} \Rightarrow \mathbf{Alg}(F)$ is locally presentable.

$\mathbf{Mnd}(\mathbb{D})$ is a full subcategory of $\mathbf{Alg}(F)$ closed under limits and filtered colimits. Thus, by the Reflection Theorem, $\mathbf{Mnd}(\mathbb{D})$ is also locally presentable. $\square$

The above theorem *cannot* be dualized.

■ Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

Suppose $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits. If furthermore, $\mathbb{D}$ has stable local initial objects, then $\text{Cmd}(\mathbb{D})$ is locally presentable.

Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

Suppose $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits. If furthermore, $\mathbb{D}$ has stable local initial objects, then $\text{Cmd}(\mathbb{D})$ is locally presentable.

Proof. Consider the F -coalgebras for the endofunctor

$$\begin{aligned} F : \text{End}(\mathbb{D}) &\rightarrow \text{End}(\mathbb{D}) \\ C_X &\mapsto (C_X \odot C_X) \times 1_X, \end{aligned}$$

that is some $C_X : X \rightarrow X$ in $\text{End}(\mathbb{D})$ equipped with a structure map

$$\gamma : C_X \Rightarrow FC_X \iff \gamma : C_X \Rightarrow (C_X \odot C_X) \times 1_X \iff \begin{cases} \gamma_1 : C_X \Rightarrow C_X \odot C_X \\ \gamma_2 : C_X \Rightarrow 1_X \end{cases}$$

Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

Suppose $\mathbb{D}$ is locally presentable and the functors $-\odot-$ and $1_{(-)}$ preserve filtered colimits. If furthermore, $\mathbb{D}$ has stable local initial objects, then $\text{Cmd}(\mathbb{D})$ is locally presentable.

Proof. Consider the F -coalgebras for the endofunctor

$$\begin{aligned} F : \text{End}(\mathbb{D}) &\rightarrow \text{End}(\mathbb{D}) \\ C_X &\mapsto (C_X \odot C_X) \times 1_X, \end{aligned}$$

that is some $C_X : X \rightarrow X$ in $\text{End}(\mathbb{D})$ equipped with a structure map

$$\gamma : C_X \Rightarrow FC_X \iff \gamma : C_X \Rightarrow (C_X \odot C_X) \times 1_X \iff \begin{cases} \gamma_1 : C_X \Rightarrow C_X \odot C_X \\ \gamma_2 : C_X \Rightarrow 1_X \end{cases}$$

F is finitary
 $\text{End}(\mathbb{D})$ is locally presentable since $\mathbb{D}$ is $\left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \text{Coalg}(F)$ is locally presentable.

■ **Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)**

Suppose $\mathbb{D}$ is locally presentable and the functors $-\odot -$ and $1_{(-)}$ preserve filtered colimits. If furthermore, $\mathbb{D}$ has stable local initial objects, then $\text{Cmd}(\mathbb{D})$ is locally presentable.

Proof. Consider the F -coalgebras for the endofunctor

$$F : \text{End}(\mathbb{D}) \rightarrow \text{End}(\mathbb{D})$$

$$C_X \mapsto (C_X \odot C_X) \times 1_X,$$

that is some $C_X : X \rightarrow X$ in $\text{End}(\mathbb{D})$ equipped with a structure map

$$\gamma : C_X \Rightarrow FC_X \iff \gamma : C_X \Rightarrow (C_X \odot C_X) \times 1_X \iff \begin{cases} \gamma_1 : C_X \Rightarrow C_X \odot C_X \\ \gamma_2 : C_X \Rightarrow 1_X \end{cases}$$

F is finitary
 $\text{End}(\mathbb{D})$ is locally presentable since $\mathbb{D}$ is $\left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \text{Coalg}(F)$ is locally presentable.

$\text{Cmd}(\mathbb{D})$ is contained in $\text{Coalg}(F)$ but for F -coalgebras, their structure maps are neither globular nor do they need to satisfy any coassociativity and counitality conditions.

First we *fix the globularity of the structure cells*.

Consider the full subcategory $\overline{\text{Coalg}(F)}$ of $\text{Coalg}(F)$ spanned by F -coalgebras with *globular* structure maps.

First we *fix the globularity of the structure cells*.

Consider the full subcategory $\overline{\text{Coalg}(F)}$ of $\text{Coalg}(F)$ spanned by F -coalgebras with *globular* structure maps.

$\overline{\text{Coalg}(F)}$ is closed under all colimits, thus cocomplete
 $\overline{\text{Coalg}(F)}$ is closed under all subobjects (!) hence accessible $\left. \vphantom{\overline{\text{Coalg}(F)}} \right\} \Rightarrow \overline{\text{Coalg}(F)}$ is locally presentable.

First we *fix the globularity of the structure cells*.

Consider the full subcategory $\overline{\text{Coalg}(F)}$ of $\text{Coalg}(F)$ spanned by F -coalgebras with *globular* structure maps.

$\overline{\text{Coalg}(F)}$ is closed under all colimits, thus cocomplete
 $\overline{\text{Coalg}(F)}$ is closed under all subobjects (!) hence accessible $\left. \vphantom{\overline{\text{Coalg}(F)}} \right\} \Rightarrow \overline{\text{Coalg}(F)}$ is locally presentable.

Now we *force coassociativity and counitality axioms*.

$\text{Cmd}(\mathbb{D})$ can be described as an equifier

$$\begin{array}{ccccc}
 & U & & U & \\
 \text{Coalg}(F) & \xrightarrow{\quad} & \text{End}(\mathbb{D}) & \text{Coalg}(F) & \xrightarrow{\quad} & \text{End}(\mathbb{D}) & \text{Coalg}(F) & \xrightarrow{\quad} & \text{End}(\mathbb{D}) \\
 \downarrow \phi_1, \psi_1 & & & \downarrow \phi_2, \psi_2 & & & \downarrow \phi_3, \psi_3 & & \\
 U & \xrightarrow{\quad} & \text{End}(\mathbb{D}) & U & \xrightarrow{\quad} & \text{End}(\mathbb{D}) & U & \xrightarrow{\quad} & \text{End}(\mathbb{D}) \\
 & \nearrow \odot^3 & & \nearrow - \odot 1_{(-)} & & & \nearrow 1_{(-)} \odot - & &
 \end{array}$$

All functors are accessible, therefore $\text{Cmd}(\mathbb{D})$ is accessible and cocomplete, hence locally presentable. $\square$

REFERENCES

-  J. Adámek, J. Rosický, *Locally presentable and Accessible Categories*, London Mathematical Society Lecture Note Series 189, Cambridge University Press.
-  V. Aravantinos-Sotiropoulos, T. Tsantilas, C. Vasilakopoulou, *On categories of monads and comonads in double categories*, arXiv:2606.26755.
-  V. Aravantinos-Sotiropoulos, C. Vasilakopoulou, *Sweedler theory for double categories*, arXiv:2408.03180.
-  H.-E. Porst, *On categories of monoids, comonoids, and bimonoids*, Quaest. Math., 31(2):127–139, 2008.
-  M. Shulman, *Framed bicategories and monoidal fibrations*, Theory and Applications of Categories, 20(18):650–738, 2008.

Thank you for your attention!