

From group gradings to 2-group gradings: categories and invariants of flat 2-bundles

The 113th Peripatetic Seminar on Sheaves and Logic

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Homotopy n -types

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A *crossed module* is a group homomorphism $\chi : E \rightarrow H$ with H acts on E (denoted $h \cdot e = {}^h e$ for $h \in H$ and $e \in E$) such that

- χ is H -equivariant (H acts on itself by conjugation) i.e.
 $\chi({}^h e) = h\chi(e)h^{-1}$ for all $h \in H$ and $e \in E$
- χ satisfies Peiffer identity, i.e. $\chi(e)e' = ee'e^{-1}$ for all $e, e' \in E$.

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Example: For any pair (X, A) of pointed topological spaces, the homotopy boundary map $\pi_2(X; A, x) \rightarrow \pi_1(A, x)$ is a crossed module.

Graded (Fusion) Categories

Monoidal Categories and Graded Point of View

A **monoidal category** is a category $\mathcal{C}$ equipped with

- a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$;
- a unit object $\mathbb{1} \in \mathcal{C}$;
- natural isomorphisms

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \quad \mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1},$$

satisfying the usual coherence conditions.

Throughout, we work with $\mathbb{C}$ -linear categories.

Graded Generalization (Idea)

Monoidal Category $\leftrightarrow \{\bullet\}$

G -graded Monoidal Category $\leftrightarrow K(G, 1)$ -space BG

χ -graded Monoidal Category $\leftrightarrow B\chi$ where $\chi: E \rightarrow H$

Group Graded Monoidal Categories

Let H be a group. An *H-graded category* is a $\mathbb{C}$ -linear monoidal category $(\mathcal{C}, \otimes)$ endowed with a family $\{\mathcal{C}_x\}_{x \in H}$ of full subcategories such that

- $X = \bigoplus_{i=1}^n X_i$ where $X_i \in \mathcal{C}_{x_i}$ for any object $X \in \mathcal{C}$,
- $\text{Hom}_{\mathcal{C}}(X, Y) = 0$ when $X \in \mathcal{C}_x$ and $Y \in \mathcal{C}_y$ with $x \neq y$,
- $X \otimes Y \in \mathcal{C}_{xy}$ when $X \in \mathcal{C}_x$ and $Y \in \mathcal{C}_y$,
- $\mathbb{1} \in \mathcal{C}_1$.

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Example: Consider the category Vec_H whose

- objects are $V = \bigoplus_{x \in H} V_x$ fin.-dim. H -graded $\mathbb{C}$ -vector spaces
- morphisms are linear maps $f: V \rightarrow W$ with $f(V_x) \subseteq W_x \quad \forall x \in H$
- monoidal product is the usual tensor product, graded by

$$(V \otimes W)_z = \bigoplus_{xy=z} V_x \otimes W_y;$$

- monoidal unit is $\mathbb{C}$ with degree 1.

Crossed Module Graded Monoidal Categories

Let $\chi : E \rightarrow H$ be a crossed module. A χ -graded category is a $\mathbb{C}$ -linear monoidal category $(\mathcal{C}, \otimes)$ which is

- E -Hom graded, i.e. $\text{Hom}_{\mathcal{C}}(X, Y) = \oplus_{e \in E} \text{Hom}_{\mathcal{C}}^e(X, Y)$ for all $X, Y \in \mathcal{C}$.
- endowed with a subclass $\mathcal{C}_{\text{hom}}$ and a degree map $|\cdot| : \mathcal{C}_{\text{hom}} \rightarrow H$ such that
 - $X = \oplus_{i=1}^n X_i$ where $X_i \in \mathcal{C}_{\text{hom}}$,
 - For $X, Y \in \mathcal{C}_{\text{hom}}$, we have $\text{Hom}_{\mathcal{C}}^e(X, Y) = 0$ if $|Y| \neq \chi(e)|X|$,
 - For $X, Y \in \mathcal{C}_{\text{hom}}$, we have $X \otimes Y = \oplus_{i=1}^n Z_i$ with $|Z_i| = |X||Y|$,
 - $|1| = 1 \in H$,
 - For any homogeneous morphisms α, β with $s(\alpha) \in \mathcal{C}_{\text{hom}}$, we have $|\alpha \otimes \beta| = |\alpha| \left(|s(\alpha)| |\beta| \right) \in E$,

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Example: Consider the category $\mathbb{C}\mathcal{G}_{\chi}$ whose

- $\text{Ob}(\mathbb{C}\mathcal{G}_{\chi}) = H$
- $\text{Hom}_{\mathbb{C}\mathcal{G}_{\chi}}(x, y) = \{e \in E \mid y = \chi(e)x\}\mathbb{C}$ for $x, y \in H$.
- monoidal product of objects $x \otimes y = xy$
- monoidal product of morphisms $(x \xrightarrow{e} y) \otimes (z \xrightarrow{f} t) = xz \xrightarrow{e^x f} yt$.

Theorem (S.-Virelizier)

Let $(A = \{A_x\}_{x \in H}, \phi)$ be a Hopf χ -coalgebra over an algebraically closed field of characteristic zero such that

- A_1 is semisimple,*
- A_x is nonzero and finite dimensional for all $x \in H$,*

then the category $\text{mod}_\chi(A)$ is χ -fusion.

Producing Examples

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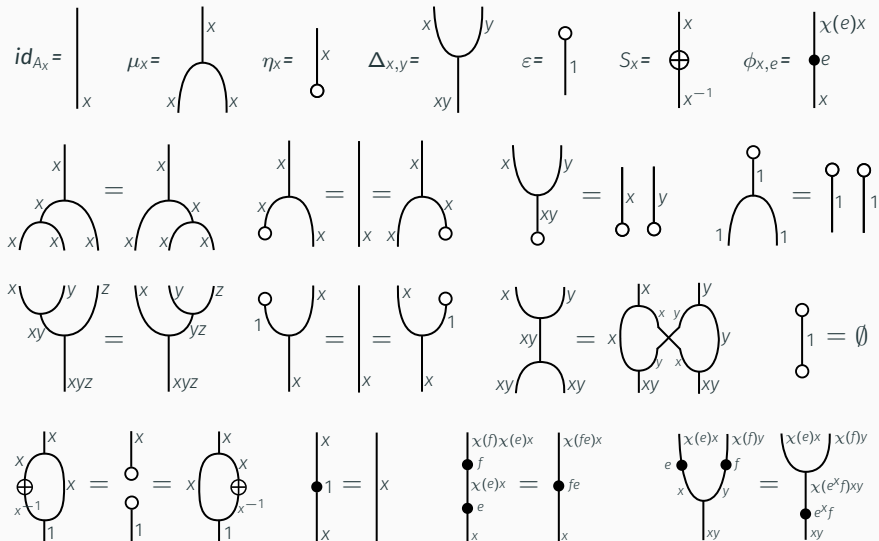
A Hopf χ -coalgebra is a collection of $\mathbb{k}$ -algebras $\{A_x\}_{x \in H}$ with

- Comultiplications $\Delta_{x,y}: A_{xy} \rightarrow A_x \otimes A_y$ which are coassociative,
- Counit $\varepsilon: A_1 \rightarrow \mathbb{k}$ which is counital,
- Antipodes $S_x: A_{x^{-1}} \rightarrow A_x$ satisfying antipode axioms.

And a family of algebra isomorphisms $\phi_{x,e}: A_x \rightarrow A_{\chi(e)x}$ satisfying

- Identity $\phi_{x,1} = \text{id}_{A_x}$,
- Action law $\phi_{x,ef} = \phi_{\chi(f)x,e} \circ \phi_{x,f}$,
- Comultiplication compatib. $(\phi_{x,e} \otimes \phi_{y,f}) \circ \Delta_{x,y} = \Delta_{\chi(e)x, \chi(f)y} \circ \phi_{xy,fe}$,
- Antipode compatibility $\phi_{x,e} \circ S_x = S_{\chi(e)x} \circ \phi_{x^{-1},e^{-1}}$.

Hopf χ -coalgebras: Graphical Definition

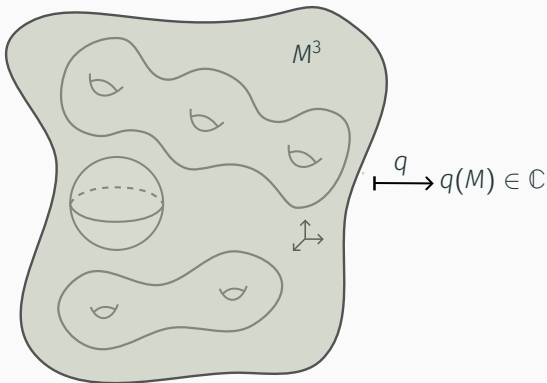


Applications: Generalizing Quantum Invariants

Quantum Invariants of 3-manifolds

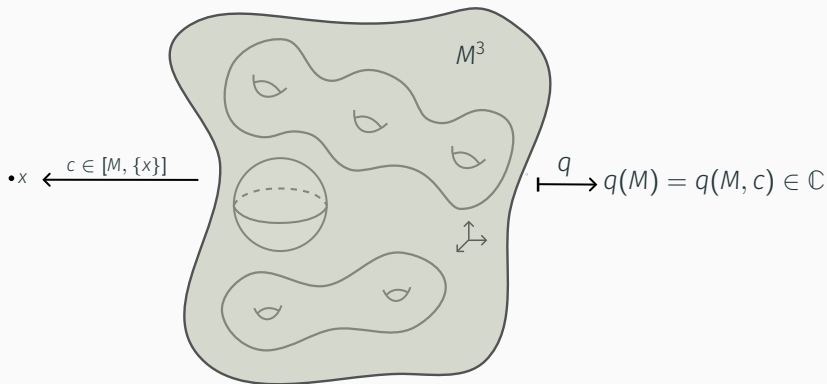
Let M be a closed connected oriented 3-dimensional manifold and let q be a (numerical) quantum invariant of 3-manifolds such as

- Turaev-Viro-Barrett-Westbury state-sum invariant
- Kuperberg invariant



Another Perspective: q is invariant of a map

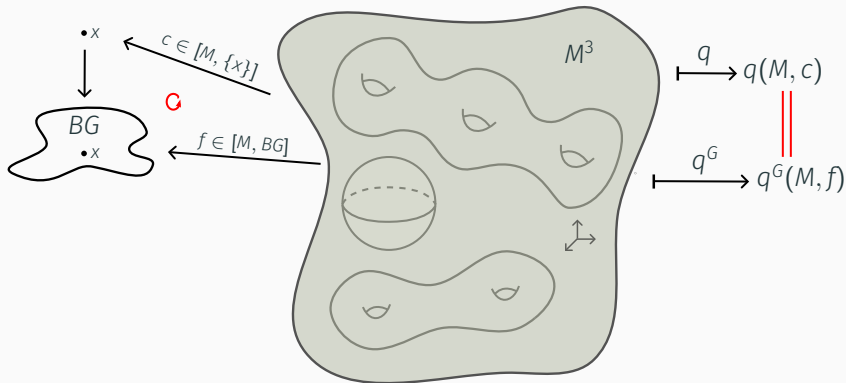
- Recall that a connected 0-type is a singleton $\{x\}$.
- $[M, \{x\}]$ is also a singleton.
- There is no harm in viewing $q(M)$ as an invariant of the constant map $c: M \rightarrow \{x\}$. (or invariant of the trivial group bundle over M)



Generalization from 0-type to 1-type

From this perspective, the generalization of the quantum invariant q to homotopy classes of maps to a connected 1-type BG is an assignment q^G that

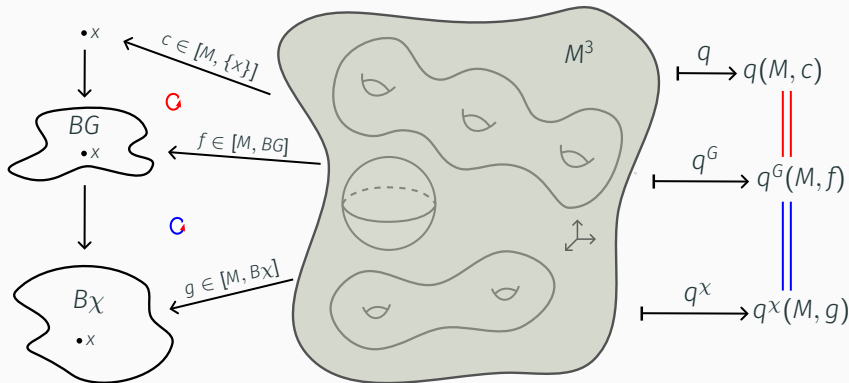
- assigns $q^G(M, f) \in \mathbb{C}$ for every $f \in [M, BG]$,
- restricts to $q(M, [c])$ when $c: M \rightarrow BG$ is the constant map.



Generalization from 1-type to 2-type

Similarly, the generalization of the quantum invariant q^G to homotopy classes of maps to a connected 2-type $B\chi$ with $\pi_1(B\chi) = G$ is an assignment q^χ that

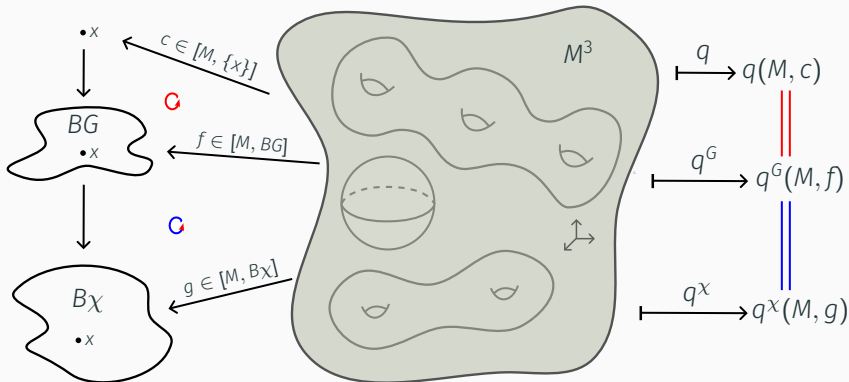
- assigns $q^\chi(M, g) \in \mathbb{C}$ for every $g \in [M, B\chi]$,
- restricts to $q^G(M, f) \in \mathbb{C}$ whenever g lifts to a map $f: M \rightarrow BG$.



Examples of q^x

For any crossed module $\chi: E \rightarrow H$, examples of q^x include

- S.-Virelizier **state-sum** homotopy quantum field theory (HQFT) derived from spherical χ -fusion categories,
- S.-Virelizier's extension of **Kuperberg invariant** using involutory Hopf χ -coalgebras.



Thank you for your attention!