

SKEW STRUCTURES AS MULTI ACTIONS

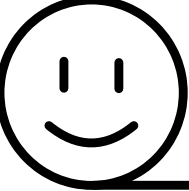
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Tallinn University of Technology

(j.w.w. Nathanael Arkor)

27.9.2026

PSSL 113, Louvain-la-Neuve



TALK STRUCTURE

- ① What are skew monoidal categories?
- ② How do we get skew monoidal cat-s in practice?
- ③ Detour: representable multiactions
- ④ Skew structures as multiactions

1 SKEW MONOIDAL CATEGORIES

Def. A **SKEW MONOIDAL CATEGORY** is a category $\mathcal{C}$ together with:

▷ tensor $\otimes: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
 $(a, b) \longmapsto a \otimes b$

▷ unit $i \in \mathcal{C}$

▷ associativity & unitality up to ~~isomorphism~~

$$\alpha_{a,b,c}: (a \otimes b) \otimes c \xrightarrow{\quad} a \otimes (b \otimes c)$$

$$\rho_a: a \xrightarrow{\quad} a \otimes i$$

$$\lambda_a: i \otimes a \xrightarrow{\quad} a$$

+ 5 coherence axioms

EXAMPLES

- $J: \mathcal{C} \longrightarrow \mathcal{D}$ functor
s.t. $\forall F \in [\mathcal{C}, \mathcal{D}]: \text{Lan}_J F$ exists

$[\mathcal{C}, \mathcal{D}]$ skew monoidal with

$$F \otimes G = \text{Lan}_J F \circ G \quad \& \quad \text{unit } J$$

- $B \in \text{Vect}_{\mathbb{K}}$ bialgebra

$\text{Vect}_{\mathbb{K}}$ skew monoidal with

$$X \hat{\otimes} Y := X \otimes B \otimes Y \quad \& \quad \text{unit } B$$

② SKEW MONOIDAL CATS & ACTIONS

Common theme in constructing examples of skew monoidal categories:
ACTIONS

- Strong actegory $\mathcal{A}$ of (skew) monoidal category $\mathcal{V}$

& adjoint pair

$$\mathcal{V} \begin{array}{c} \xrightarrow{j*(-)} \\ \xleftarrow[\jmath]{\perp} \end{array} \mathcal{A}$$

[Szlachányi 2016]

[Procházková 2025]

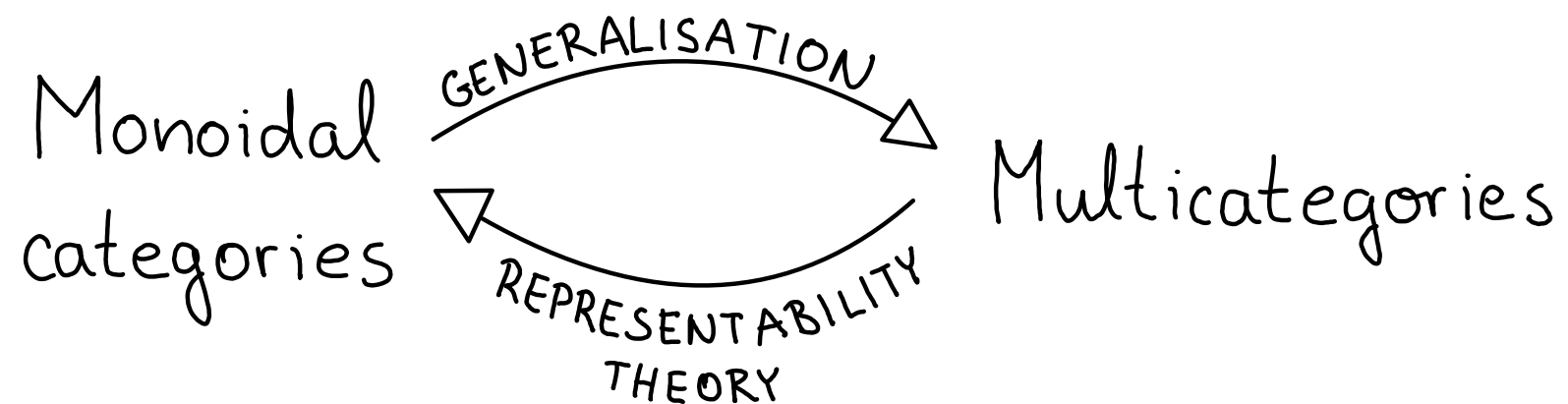
- Skew $\mathcal{V}$ -actegory $\mathcal{A}$ & strong adjunction $\mathcal{V} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow[\eta]{\perp} \end{array} \mathcal{A}$

[Fiore & Saville - CT 2018 talk]

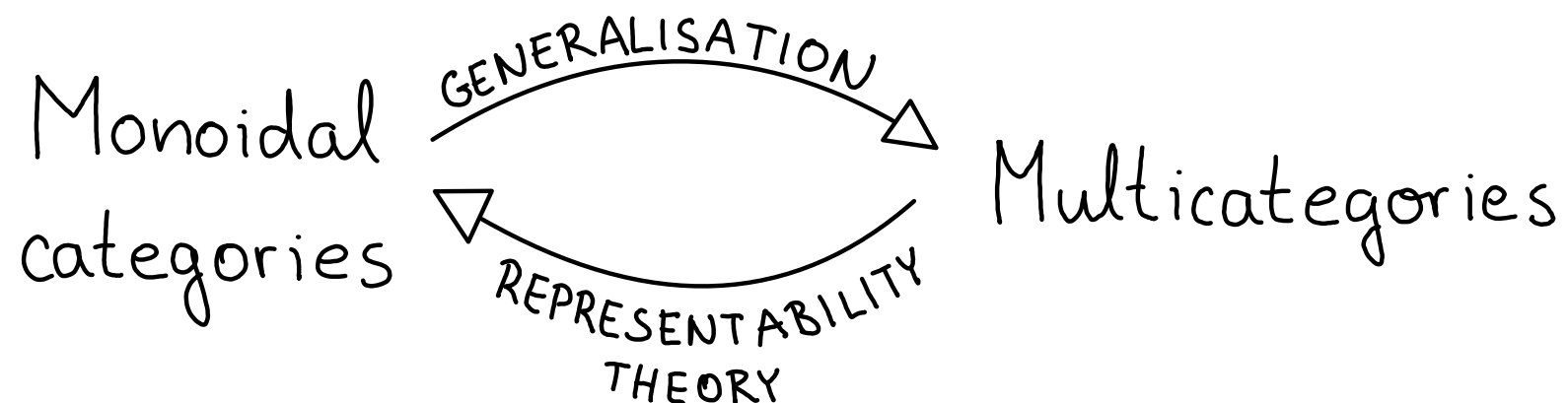
- Skew $\mathcal{V}$ -actegory $\mathcal{A}$ & skew warping $K: \mathcal{A} \longrightarrow \mathcal{V}$

[Lack & Street 2015]

3* REPRESENTABLE MULTICATEGORIES



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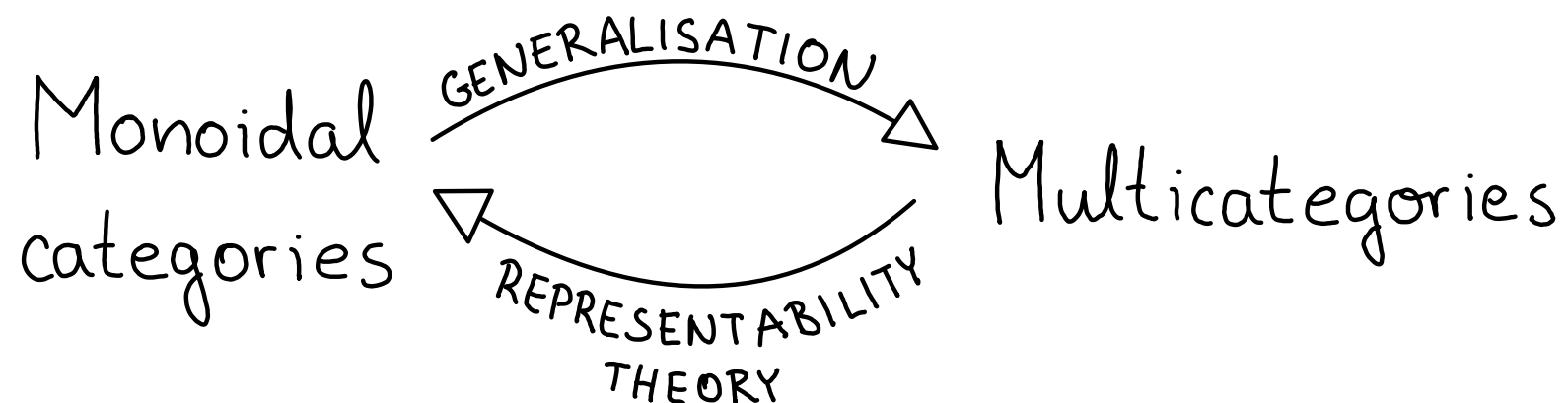


Def A multicategory M is **REPRESENTABLE** if for every $a_1, \dots, a_k \in |M|$ there is

- ▷ an object $a_1 \otimes \dots \otimes a_k$ and
- ▷ a **natural isomorphism**

$$M(b_1, \dots, b_{i-1}, a_1 \otimes \dots \otimes a_k, b_{i+1}, \dots, b_n; c) \cong M(b_1, \dots, b_{i-1}, a_1, \dots, a_k, b_{i+1}, \dots, b_n; c)$$

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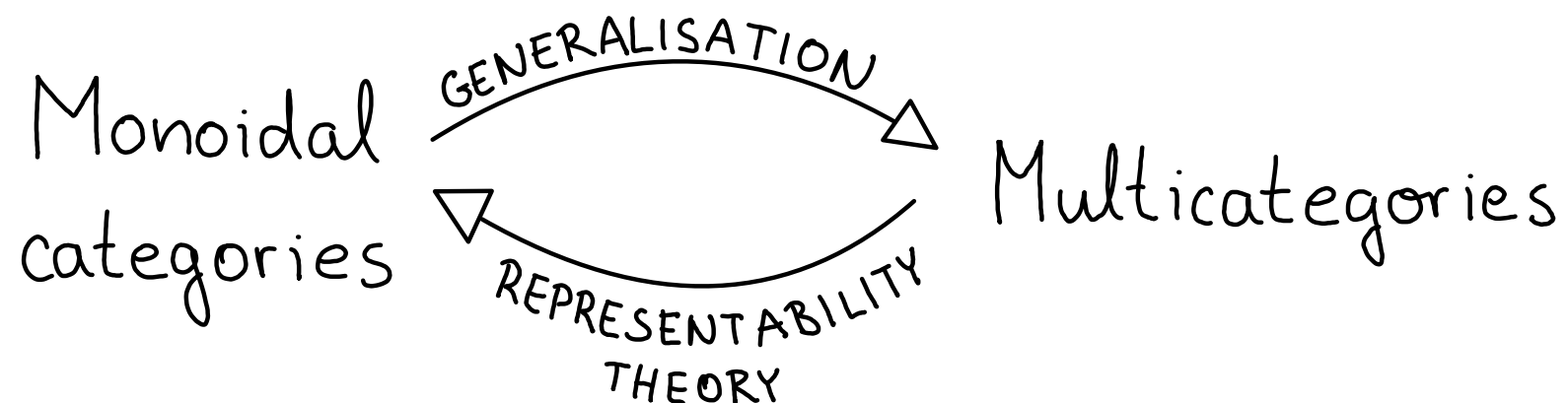
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INTUITION: $M(a_1, a_2, a_3, \dots, a_{n-1}, a_n; b)$

M is representable $\approx$

$\approx$ I can turn any of the commas **,** into tensors $\otimes$

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M is representable $\approx$

$\approx$ I can turn any of the commas $,$ into tensors $\otimes$

[Hermida, 2000]

Theorem

There is a 2-equivalence

$$\text{MonCat} \xrightleftharpoons{\cong} \text{Multicat}_{\text{rep.}}$$

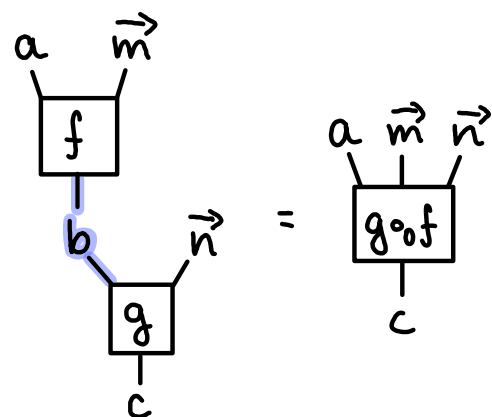
3 MULTIACTEGORIES

M ... multicategory

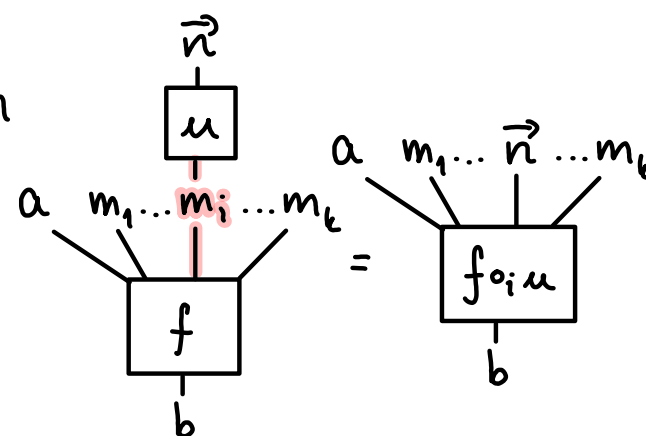
Def. An M -multiactegory A consists of $\triangleright$ objects $|A|$

$\triangleright$ for each $a, b \in |A|$ and each list $m_1, \dots, m_k \in |M|$, $k \geq 0$
a set of **multimaps** $A(a, m_1, \dots, m_k; b)$

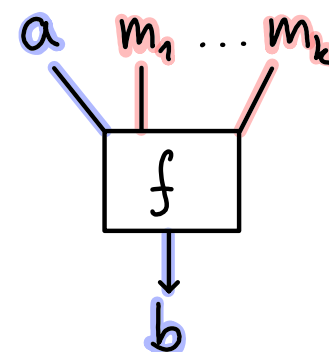
A-composition



M-composition



associative
& unital



EXAMPLES

$\mathcal{C}$... category

$$\mathcal{C} \times [\mathcal{C}, \mathcal{C}] \xrightarrow{\text{ev}} \mathcal{C}$$

$$(a, F) \longmapsto Fa$$

$\hookrightarrow$ action of a monoidal category

$\mathcal{D} \subseteq [\mathcal{C}, \mathcal{C}]$... subcategory
(not necessarily monoidal)

$$\mathcal{C} \times \mathcal{D} \xrightarrow{\text{ev}} \mathcal{C}$$

$$(a, F) \longmapsto Fa$$

③ REPRESENTABLE MULTIACTEGORIES

A : M -multiaction

$$A(a, \underset{\text{I}}{m_1}, \dots, \underset{\text{II}}{m_k}; b) \quad M(\underset{\text{III}}{m_1}, \dots, m_k; n)$$

we have „three different types of commas“ to turn into „tensors“

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II

$$A(a, m_1, \dots, m_i \otimes m_{i+1}, \dots, m_k; b) \cong A(a, m_1, \dots, m_i, m_{i+1}, \dots, m_k; b)$$

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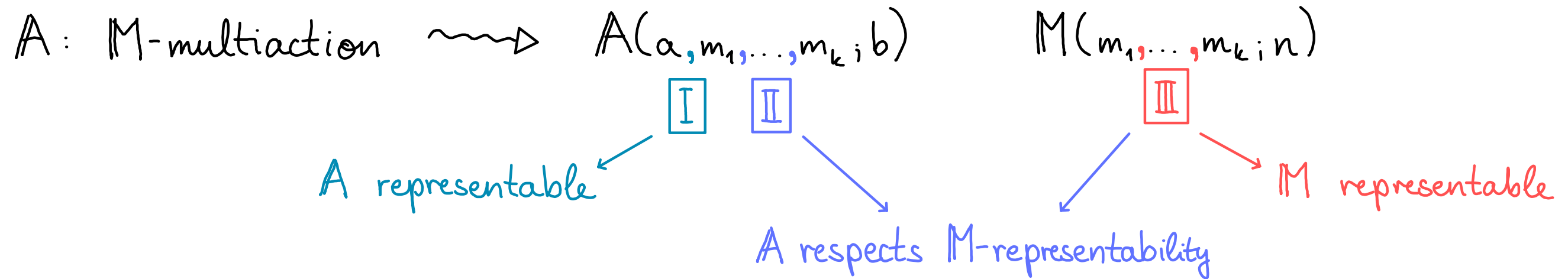
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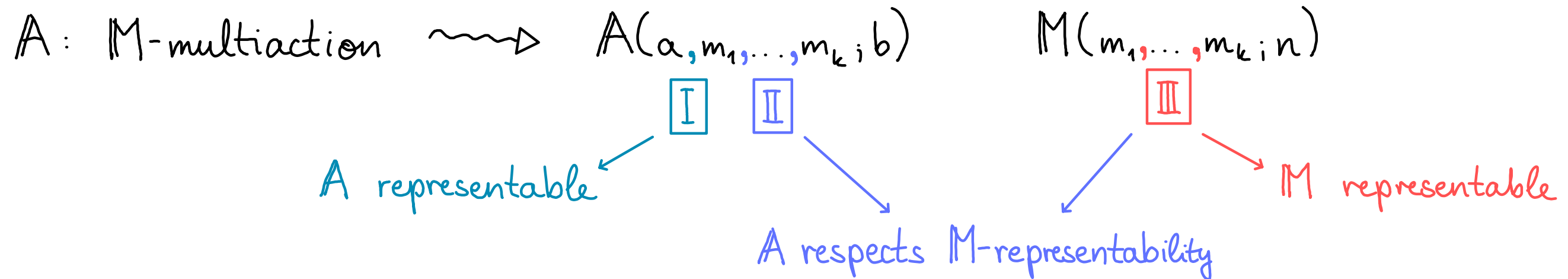
III

$$M(m_1, \dots, m_i \otimes m_{i+1}, \dots, m_k; n) \cong M(m_1, \dots, m_i, m_{i+1}, \dots, m_k; n)$$

3 REPRESENTABLE MULTIACTEGORIES

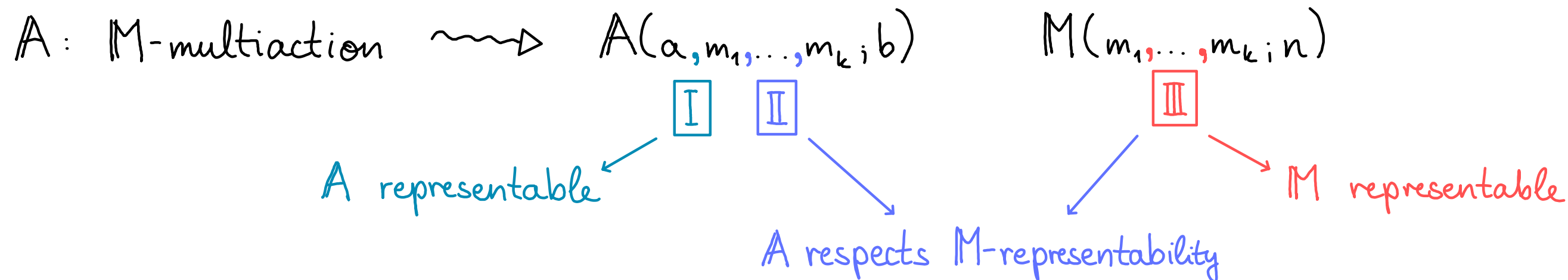


3 REPRESENTABLE MULTIACTEGORIES



Prop: I $\{A \text{ representable } \mathbb{M}\text{-multiact.}\} \simeq \left\{ \begin{array}{c} \text{A action of } FM \\ \uparrow \\ \text{free strict mon. cat. on } |\mathbb{M}| \end{array} \right\}$

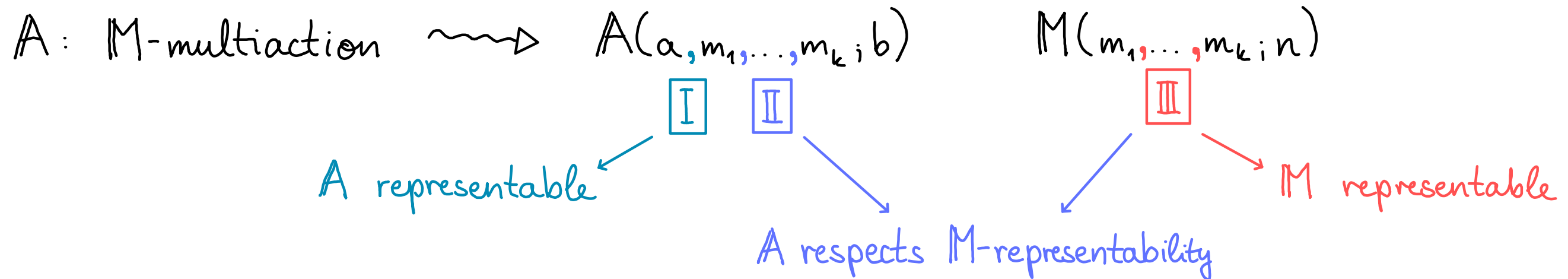
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 $\uparrow$
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Prop: $\boxed{\text{I}} + \boxed{\text{II}} + \boxed{\text{III}}$ $\left\{ \begin{array}{l} A \text{ rep. } \mathbb{M}\text{-multiact, } \mathbb{M} \text{ rep.} \\ \& A \text{ respects } \mathbb{M}\text{-rep.} \end{array} \right\} \simeq \left\{ A \text{ (strong) action of } \mathcal{M} \right\}$
 $\nwarrow$
 [Stroiński & Zorman, 2024]

4 REPRESENTING UNITS

$\mathbb{M}$ multicategory $\rightsquigarrow$ To get a unit in $\mathbb{M}$

- object $i \in |\mathbb{M}|$

- $\mathbb{M}(i, \vec{m}; n) \cong \mathbb{M}(\vec{m}; n)$ natural iso,
 $\rightsquigarrow$ nullary map $\boxed{\pi_o}_i$ in $\mathbb{M}$

"UNITS IN CATEGORIES"

$\rightsquigarrow$ PROBLEM: No nullary maps in $\mathcal{A}$.

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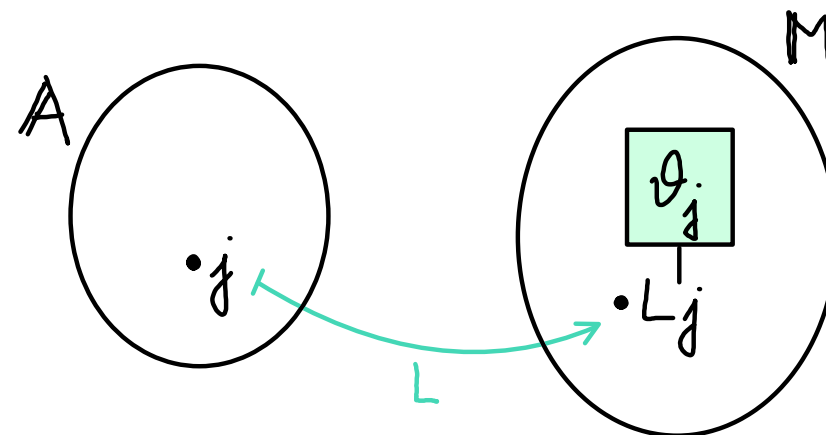
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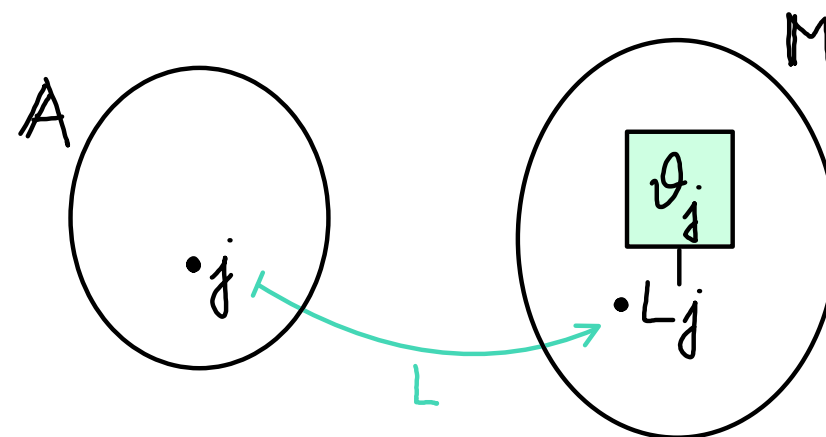
$\hookrightarrow$ Solution: Consider morphisms as well!

"NULLARY REPRESENTABILITY FOR MULTI ACTIONS"

$A \xrightarrow{L} \mathbb{M}$ $\mathbb{M}$ -multiaction morphism

$j \in |A|$ & natural isomorphism

$$A(j, m_1, \dots, m_k; b) \cong \mathbb{M}(m_1, \dots, m_n; Lb)$$

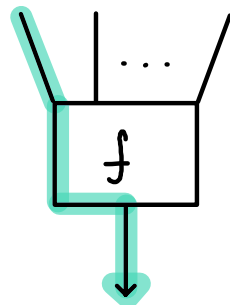


4 SKEW MULTICATEGORIES

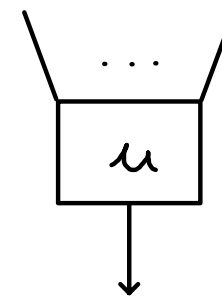
Def. A SKEW MULTICATEGORY consists of:

- ▷ multicategory M
- ▷ M -multicategory A
- ▷ identity-on-objects M -multicat. morphism $A \xrightarrow{L} M$

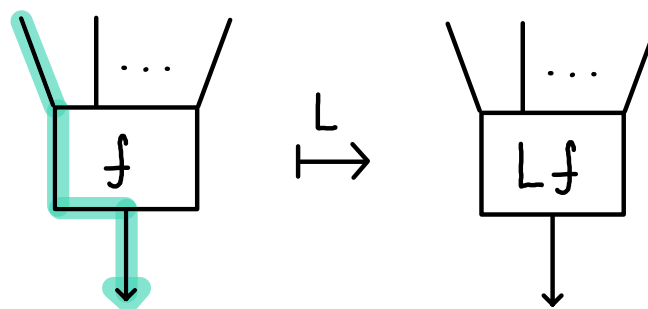
TIGHT
MAPS



LOOSE
MAPS



"LOOSENING"



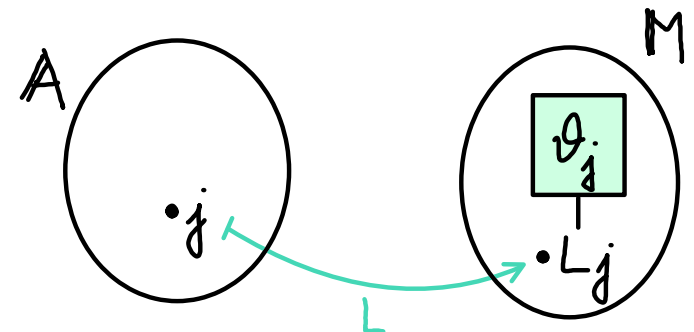
4 LEFT REPRESENTABILITY

Def. A skew multicategory $A \xrightarrow{L} M$ is (LEFT) REPRESENTABLE if

▷ A is representable as an M -multicategory

▷ there exists $j \in |A|$ and $\boxed{g_j}_{L_j} \in M$ s.t.

$$A(j, m_1, \dots, m_k; b) \cong M(m_1, \dots, m_n; Lb)$$



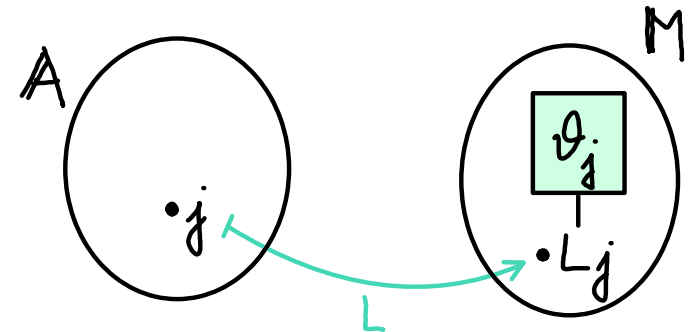
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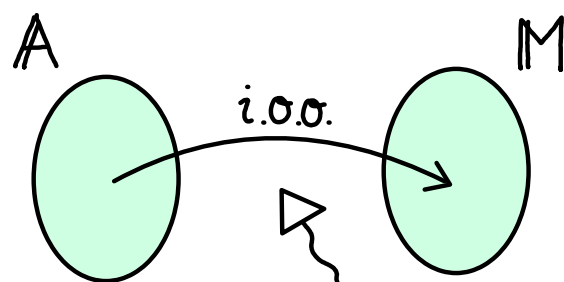
Theorem There is a 2-equivalence

$$\text{SkewMonCat} \xrightleftharpoons{\cong} \text{SkewMulticat}|_{\text{l.rep.}}$$

[Bourke & Lack, 2017]

④ MULTIACTION MORPHISMS

SKEW MULTICATEGORY

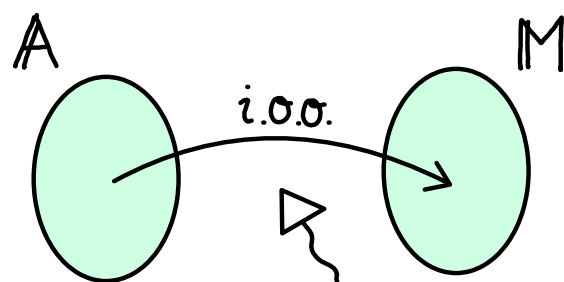


$$L|A| = |M|$$

M -multicategory morphism

④ MULTIACTION MORPHISMS

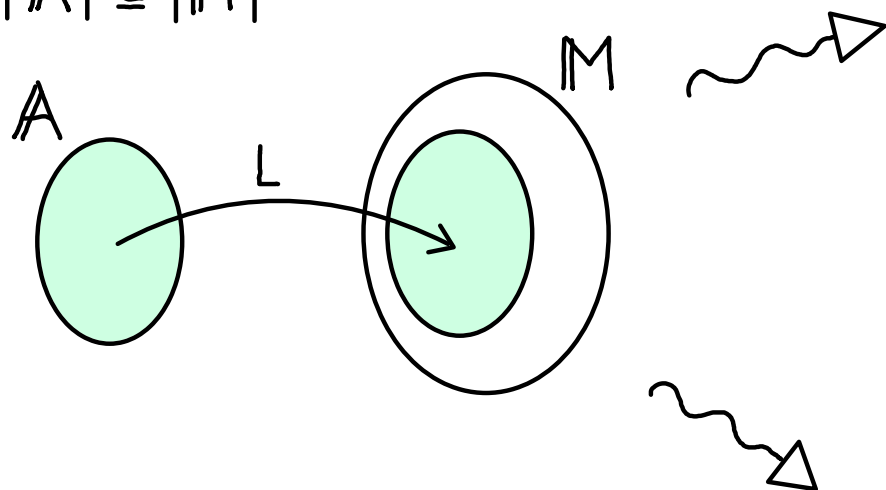
SKREW MULTICATEGORY



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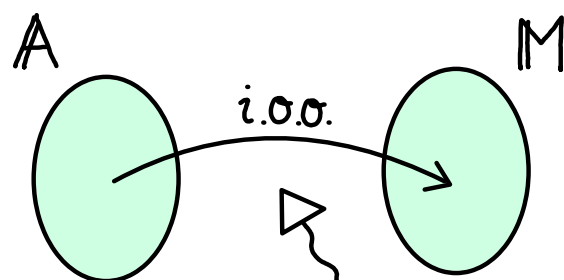
M -multiactegory morphism

$$L|A| \subseteq |M|$$



4 MULTIACTION MORPHISMS

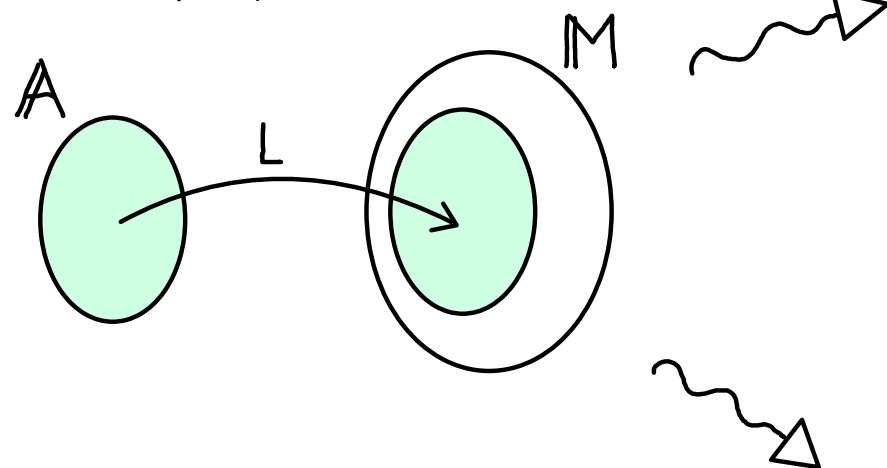
SKREW MULTICATEGORY



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M-multicategory morphism

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M-MULTIACT. MORPHISM

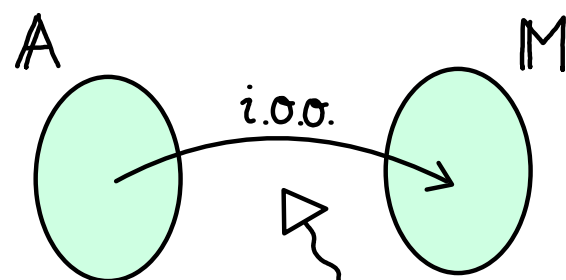
$$A(a, m_1, \dots, m_k; b)$$

$$\downarrow L$$

$$M(La, m_1, \dots, m_k; Lb)$$

4 MULTIACTION MORPHISMS

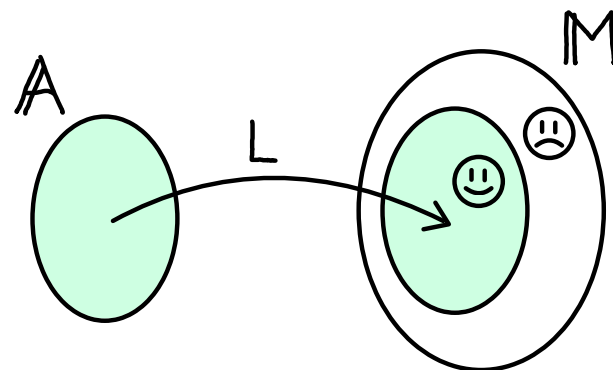
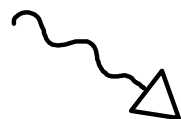
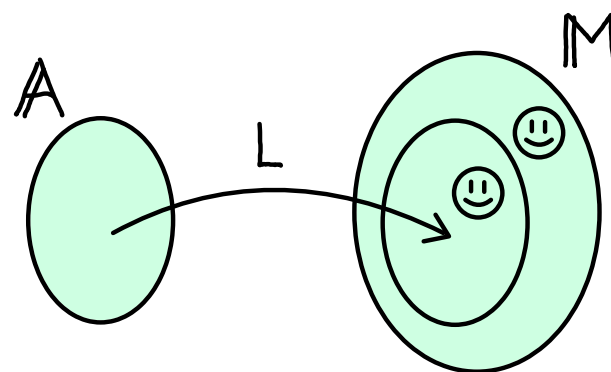
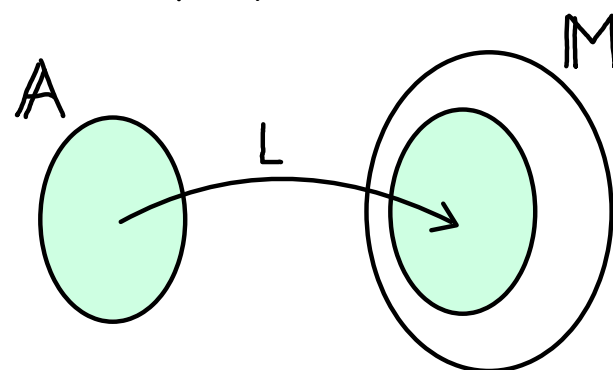
SKREW MULTICATEGORY



$$L|A| = |M|$$

M-multicategory morphism

$$L|A| \subseteq |M|$$



M-MULTIACT. MORPHISM

$$A(a, m_1, \dots, m_k; b)$$

$$\downarrow L$$

$$M(La, m_1, \dots, m_k; Lb)$$

MULTIWARPING

$$A(a, La_1, \dots, La_k; b)$$

$$\downarrow L$$

$$M(La, La_1, \dots, La_k; Lb)$$

Remark: multiwarping $\equiv$ $L|A|$ -multicategory morphism

4 LAYERING REPRESENTABILITIES

A : $\mathcal{M}$ -multiactegory $A \xrightarrow{L} \mathcal{M}$: $\mathcal{M}$ -multiact. morphism

$\mathcal{M}$ rep. multicat	A rep. multiact	L rep. multiwarp.	L rep. A -multiact. morphism	A respects $\mathcal{M}$ -rep.	
		✓			A skew monoidal category
✓	✓				A skew $\mathcal{M}$ -actegory
✓	✓			✓	A strong $\mathcal{M}$ -actegory
✓	✓	✓			A skew $\mathcal{M}$ -actegory tw. skew warping L
✓	✓	✓		✓	A strong $\mathcal{M}$ -actegory tw. skew warping L
✓	✓		✓		A skew $\mathcal{M}$ -actegory tw. $j*(-) \dashv L$
✓	✓		✓	✓	A strong $\mathcal{M}$ -actegory tw. $j*(-) \dashv L$

4 SKEW MULTIACTEGORIES

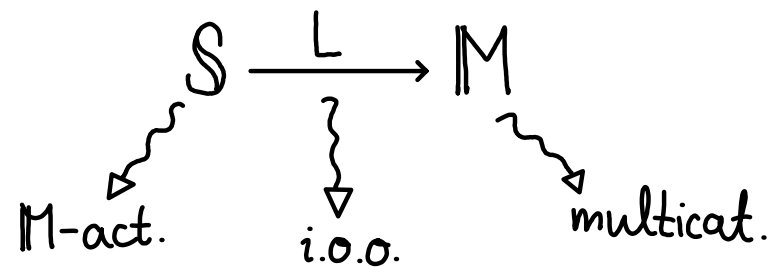
$A: M\text{-multiactegory}$ $A \xrightarrow{L} M: M\text{-multiact. morphism}$

↳ taking M representable, we can capture

(SKEW) ACTIONS OF MONOIDAL CATEGORIES

How do we get (skew) actions of SKEW MONOIDAL categories?

Skew multicategory:



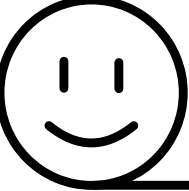
SKEW MULTIACTION
of $S \xrightarrow{L} M$ is
! an M -multiactegory

Theorem

There is a 2-equivalence

$$\text{SkewAct} \xrightleftharpoons{\cong}$$

Skew Multiact $\left| \begin{array}{l} S \xrightarrow{L} M \text{ l. rep.} \\ A \text{ rep.} \end{array} \right.$



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