

Derived torsion theories in a non-pointed context

Andrea Montoli

Università degli Studi di Milano

PSSL 113, Louvain-la-Neuve

Joint work with Andrea Cappelletti

Dedicated to the memory of Andrée Ehresmann

The notion of torsion theory has been introduced by Dickson in abelian categories

The notion of torsion theory has been introduced by Dickson in abelian categories and then generalized by Bourn and Gran to the context of homological categories.

The notion of torsion theory has been introduced by Dickson in abelian categories and then generalized by Bourn and Gran to the context of homological categories.

Definition (Borceux-Bourn)

An homological category is a pointed, regular, Bourn-protomodular category

The notion of torsion theory has been introduced by Dickson in abelian categories and then generalized by Bourn and Gran to the context of homological categories.

Definition (Borceux-Bourn)

An homological category is a pointed, regular, Bourn-protomodular category (= split short five lemma).

The notion of torsion theory has been introduced by Dickson in abelian categories and then generalized by Bourn and Gran to the context of homological categories.

Definition (Borceux-Bourn)

An homological category is a pointed, regular, Bourn-protomodular category (= split short five lemma).

In homological categories the classical homological lemmas are valid: the short five lemma, the 9 lemma, the snake lemma, the Noether's isomorphisms theorems...

Definition

A torsion theory in a pointed category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

Definition

A torsion theory in a pointed category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- 1 $\text{Hom}(T, F) = \{0\} \quad \forall T \in \mathcal{T}, F \in \mathcal{F};$

Definition

A torsion theory in a pointed category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- 1 $\text{Hom}(T, F) = \{0\} \quad \forall T \in \mathcal{T}, F \in \mathcal{F};$
- 2 for all $X \in \mathcal{C}$ there is a short exact sequence

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

with $T \in \mathcal{T}, F \in \mathcal{F}$.

Definition

A torsion theory in a pointed category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- 1 $\text{Hom}(T, F) = \{0\} \quad \forall T \in \mathcal{T}, F \in \mathcal{F};$
- 2 for all $X \in \mathcal{C}$ there is a short exact sequence

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

with $T \in \mathcal{T}, F \in \mathcal{F}$.

The short exact sequence is unique up to isomorphisms,

Definition

A torsion theory in a pointed category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- 1 $\text{Hom}(T, F) = \{0\} \quad \forall T \in \mathcal{T}, F \in \mathcal{F};$
- 2 for all $X \in \mathcal{C}$ there is a short exact sequence

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

with $T \in \mathcal{T}, F \in \mathcal{F}$.

The short exact sequence is unique up to isomorphisms, $\mathcal{T}$ is monoreflective,

Definition

A torsion theory in a pointed category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- 1 $\text{Hom}(T, F) = \{0\} \quad \forall T \in \mathcal{T}, F \in \mathcal{F};$
- 2 for all $X \in \mathcal{C}$ there is a short exact sequence

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

with $T \in \mathcal{T}, F \in \mathcal{F}$.

The short exact sequence is unique up to isomorphisms, $\mathcal{T}$ is monoreflective, $\mathcal{F}$ is epi-reflective.

Let $\mathcal{N}$ be an ideal of morphisms in the sense of C. Ehresmann.

Let $\mathcal{N}$ be an ideal of morphisms in the sense of C. Ehresmann.

The $\mathcal{N}$ -kernel of $f: A \rightarrow B$ is the universal arrow $k: K \rightarrow A$ such that $fk \in \mathcal{N}$.

Non-pointed torsion theories

Let $\mathcal{N}$ be an ideal of morphisms in the sense of C. Ehresmann.

The $\mathcal{N}$ -kernel of $f: A \rightarrow B$ is the universal arrow $k: K \rightarrow A$ such that $fk \in \mathcal{N}$.

The definition of $\mathcal{N}$ -cokernel is dual.

Non-pointed torsion theories

Let $\mathcal{N}$ be an ideal of morphisms in the sense of C. Ehresmann.

The $\mathcal{N}$ -kernel of $f: A \rightarrow B$ is the universal arrow $k: K \rightarrow A$ such that $fk \in \mathcal{N}$.

The definition of $\mathcal{N}$ -cokernel is dual.

Definition

A short $\mathcal{N}$ -exact sequence is a diagram

$$K \xrightarrow{k} A \xrightarrow{f} B$$

where $k = \mathcal{N}\text{-ker}(f)$ and $f = \mathcal{N}\text{-coker}(k)$.

Non-pointed torsion theories

Let $\mathcal{N}$ be an ideal of morphisms in the sense of C. Ehresmann.

The $\mathcal{N}$ -kernel of $f: A \rightarrow B$ is the universal arrow $k: K \rightarrow A$ such that $fk \in \mathcal{N}$.

The definition of $\mathcal{N}$ -cokernel is dual.

Definition

A short $\mathcal{N}$ -exact sequence is a diagram

$$K \xrightarrow{k} A \xrightarrow{f} B$$

where $k = \mathcal{N}\text{-ker}(f)$ and $f = \mathcal{N}\text{-coker}(k)$.

Moreover we suppose that the ideal $\mathcal{N}$ is closed.

Non-pointed torsion theories

Let $\mathcal{C}$ be a category with a closed ideal of morphisms $\mathcal{N}$ such that the $\mathcal{N}$ -kernels and the $\mathcal{N}$ -cokernels of identities exist.

Non-pointed torsion theories

Let $\mathcal{C}$ be a category with a closed ideal of morphisms $\mathcal{N}$ such that the $\mathcal{N}$ -kernels and the $\mathcal{N}$ -cokernels of identities exist.

Definition (Grandis-Janelidze)

A torsion theory in $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

Non-pointed torsion theories

Let $\mathcal{C}$ be a category with a closed ideal of morphisms $\mathcal{N}$ such that the $\mathcal{N}$ -kernels and the $\mathcal{N}$ -cokernels of identities exist.

Definition (Grandis-Janelidze)

A torsion theory in $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

$$\textcircled{1} \quad \text{Hom}(T, F) \subseteq \mathcal{N} \quad \forall T \in \mathcal{T}, F \in \mathcal{F};$$

Let $\mathcal{C}$ be a category with a closed ideal of morphisms $\mathcal{N}$ such that the $\mathcal{N}$ -kernels and the $\mathcal{N}$ -cokernels of identities exist.

Definition (Grandis-Janelidze)

A torsion theory in $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- ❶ $\text{Hom}(T, F) \subseteq \mathcal{N} \quad \forall T \in \mathcal{T}, F \in \mathcal{F};$
- ❷ *for all $X \in \mathcal{C}$ there is a short $\mathcal{N}$ -exact sequence*

$$T \xrightarrow{t_X} X \xrightarrow{\eta_X} F$$

with $T \in \mathcal{T}, F \in \mathcal{F}$.

Let $\mathcal{C}$ be a category with a closed ideal of morphisms $\mathcal{N}$ such that the $\mathcal{N}$ -kernels and the $\mathcal{N}$ -cokernels of identities exist.

Definition (Grandis-Janelidze)

A torsion theory in $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- 1 $\text{Hom}(T, F) \subseteq \mathcal{N} \quad \forall T \in \mathcal{T}, F \in \mathcal{F};$
- 2 for all $X \in \mathcal{C}$ there is a short $\mathcal{N}$ -exact sequence

$$T \xrightarrow{t_X} X \xrightarrow{\eta_X} F$$

with $T \in \mathcal{T}, F \in \mathcal{F}$.

Facchini-Finocchiaro-Gran: same definition, but without the existence of $\mathcal{N}$ -kernels and $\mathcal{N}$ -cokernels of identities.

Let $\mathcal{C}$ be a regular protomodular category with 0 such that the only morphism $0 \rightarrow 1$ is a regular epimorphism.

Let $\mathcal{C}$ be a regular protomodular category with 0 such that the only morphism $0 \rightarrow 1$ is a regular epimorphism. Let $\mathcal{Z}$ the full subcategory of regular quotients of 0 .

Let $\mathcal{C}$ be a regular protomodular category with 0 such that the only morphism $0 \rightarrow 1$ is a regular epimorphism. Let $\mathcal{Z}$ the full subcategory of regular quotients of 0 .

Proposition

$\mathcal{Z}$ is a posetal monoreflective subcategory of $\mathcal{C}$ and the coreflector $\mathcal{Z}$ inverts monomorphisms.

Let $\mathcal{C}$ be a regular protomodular category with 0 such that the only morphism $0 \rightarrow 1$ is a regular epimorphism. Let $\mathcal{Z}$ the full subcategory of regular quotients of 0 .

Proposition

$\mathcal{Z}$ is a posetal monoreflective subcategory of $\mathcal{C}$ and the coreflector $\mathcal{Z}$ inverts monomorphisms.

Examples: ideally exact categories (Janelidze):

Let $\mathcal{C}$ be a regular protomodular category with 0 such that the only morphism $0 \rightarrow 1$ is a regular epimorphism. Let $\mathcal{Z}$ the full subcategory of regular quotients of 0 .

Proposition

$\mathcal{Z}$ is a posetal monoreflective subcategory of $\mathcal{C}$ and the coreflector $\mathbb{Z}$ inverts monomorphisms.

Examples: ideally exact categories (Janelidze): in particular, unitary rings, Boolean algebras, Heyting algebras, MV-algebras, the duals of elementary toposes, the topological models of the mentioned varieties.

Let $\mathcal{N}_{\mathcal{Z}}$ be the ideal of morphisms that factor through $\mathcal{Z}$.

Let $\mathcal{N}_{\mathcal{Z}}$ be the ideal of morphisms that factor through $\mathcal{Z}$.
In our context, the $\mathcal{N}_{\mathcal{Z}}$ -kernels always exist

Let $\mathcal{N}_{\mathcal{Z}}$ be the ideal of morphisms that factor through $\mathcal{Z}$.
In our context, the $\mathcal{N}_{\mathcal{Z}}$ -kernels always exist and they are obtained as pullbacks

$$\begin{array}{ccc} K[f] & \longrightarrow & Z(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

Let $\mathcal{N}_{\mathcal{Z}}$ be the ideal of morphisms that factor through $\mathcal{Z}$.
In our context, the $\mathcal{N}_{\mathcal{Z}}$ -kernels always exist and they are obtained as pullbacks

$$\begin{array}{ccc} K[f] & \longrightarrow & Z(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

The $\mathcal{N}_{\mathcal{Z}}$ -cokernels do not always exist.

Let $\mathcal{N}_{\mathcal{Z}}$ be the ideal of morphisms that factor through $\mathcal{Z}$.
In our context, the $\mathcal{N}_{\mathcal{Z}}$ -kernels always exist and they are obtained as pullbacks

$$\begin{array}{ccc} K[f] & \longrightarrow & Z(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

The $\mathcal{N}_{\mathcal{Z}}$ -cokernels do not always exist. For example, in *Boole* the identity of 2×2 does not have $\mathcal{N}_{\mathcal{Z}}$ -cokernel.

Let $\mathcal{N}_{\mathcal{Z}}$ be the ideal of morphisms that factor through $\mathcal{Z}$.
In our context, the $\mathcal{N}_{\mathcal{Z}}$ -kernels always exist and they are obtained as pullbacks

$$\begin{array}{ccc} K[f] & \longrightarrow & Z(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

The $\mathcal{N}_{\mathcal{Z}}$ -cokernels do not always exist. For example, in *Boole* the identity of 2×2 does not have $\mathcal{N}_{\mathcal{Z}}$ -cokernel. Moreover, there are regular epimorphisms that are not $\mathcal{N}_{\mathcal{Z}}$ -cokernels

Let $\mathcal{N}_{\mathcal{Z}}$ be the ideal of morphisms that factor through $\mathcal{Z}$.
In our context, the $\mathcal{N}_{\mathcal{Z}}$ -kernels always exist and they are obtained as pullbacks

$$\begin{array}{ccc} K[f] & \longrightarrow & Z(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

The $\mathcal{N}_{\mathcal{Z}}$ -cokernels do not always exist. For example, in *Boole* the identity of 2×2 does not have $\mathcal{N}_{\mathcal{Z}}$ -cokernel. Moreover, there are regular epimorphisms that are not $\mathcal{N}_{\mathcal{Z}}$ -cokernels (e.g. the product projections of 2×2).

Torsion theories and factorization systems

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ satisfies condition (N) if for all

$$\mathrm{T}(K[f]) \xrightarrow{t_{K[f]}} K[f] \xrightarrow{k} A \xrightarrow{f} B,$$

where $k = \mathcal{N}_{\mathcal{Z}}\text{-ker}(f)$, $kt_{K[f]}$ is a $\mathcal{N}_{\mathcal{Z}}$ -kernel.

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ satisfies condition (N) if for all

$$\mathrm{T}(K[f]) \xrightarrow{t_{K[f]}} K[f] \xrightarrow{k} A \xrightarrow{f} B,$$

where $k = \mathcal{N}_{\mathcal{Z}}\text{-ker}(f)$, $kt_{K[f]}$ is a $\mathcal{N}_{\mathcal{Z}}$ -kernel.

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ satisfies condition (M) if, for all $\mathcal{N}_{\mathcal{Z}}$ -exact sequences

$$T \xrightarrow{k} C \xrightarrow{q} Q$$

where $T \in \mathcal{T}$, one has $F(T) \cong Z(Q)$.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (N) and (M), then the classes

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (N) and (M), then the classes

$$\mathbb{E} = \{ e \in \text{Arr}(\mathcal{C}) \mid e \text{ is a } \mathcal{N}_{\mathcal{Z}}\text{-cokernel, } K[e] \in \mathcal{T} \}$$

$$\mathbb{M} = \{ m \in \text{Arr}(\mathcal{C}) \mid K[m] \in \mathcal{F} \}$$

form a stable factorization system such that all arrows in $\mathbb{E}$ are $\mathcal{N}_{\mathcal{Z}}$ -cokernels.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (N) and (M), then the classes

$$\mathbb{E} = \{ e \in \text{Arr}(\mathcal{C}) \mid e \text{ is a } \mathcal{N}_{\mathcal{Z}}\text{-cokernel, } K[e] \in \mathcal{T} \}$$

$$\mathbb{M} = \{ m \in \text{Arr}(\mathcal{C}) \mid K[m] \in \mathcal{F} \}$$

form a stable factorization system such that all arrows in $\mathbb{E}$ are $\mathcal{N}_{\mathcal{Z}}$ -cokernels. Conversely, if $(\mathbb{E}, \mathbb{M})$ is a stable factorization system in $\mathcal{C}$ such that all arrows in $\mathbb{E}$ are $\mathcal{N}_{\mathcal{Z}}$ -cokernels, then the full subcategories

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (N) and (M), then the classes

$$\mathbb{E} = \{ e \in \text{Arr}(\mathcal{C}) \mid e \text{ is a } \mathcal{N}_{\mathcal{Z}}\text{-cokernel, } K[e] \in \mathcal{T} \}$$

$$\mathbb{M} = \{ m \in \text{Arr}(\mathcal{C}) \mid K[m] \in \mathcal{F} \}$$

form a stable factorization system such that all arrows in $\mathbb{E}$ are $\mathcal{N}_{\mathcal{Z}}$ -cokernels. Conversely, if $(\mathbb{E}, \mathbb{M})$ is a stable factorization system in $\mathcal{C}$ such that all arrows in $\mathbb{E}$ are $\mathcal{N}_{\mathcal{Z}}$ -cokernels, then the full subcategories

$$\mathcal{T} = \{ T \in \mathcal{C} \mid \exists T \rightarrow Z \in \mathbb{E} \}$$

$$\mathcal{F} = \{ F \in \mathcal{C} \mid F \rightarrow 1 \in \mathbb{M} \}$$

form a torsion theory satisfying (N) and (M).

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ satisfies condition (M') if, for all $\mathcal{N}_{\mathcal{Z}}$ -exact sequence

$$T(C) \xrightarrow{t_C} C \xrightarrow{\eta_C} F(C),$$

one has $F(T(C)) \cong Z(F(C))$.

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ satisfies condition (M') if, for all $\mathcal{N}_{\mathcal{Z}}$ -exact sequence

$$T(C) \xrightarrow{t_C} C \xrightarrow{\eta_C} F(C),$$

one has $F(T(C)) \cong Z(F(C))$. (M) implies (M') .

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ satisfies condition (M') if, for all $\mathcal{N}_{\mathcal{Z}}$ -exact sequence

$$T(C) \xrightarrow{t_C} C \xrightarrow{\eta_C} F(C),$$

one has $F(T(C)) \cong Z(F(C))$. (M) implies (M') .

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ satisfies condition (S) if, in any pullback

$$\begin{array}{ccc} T' & \xrightarrow{\chi} & Z' \\ \downarrow & & \downarrow \\ T & \xrightarrow{\eta_T} & Z \end{array}$$

where $T \in \mathcal{T}$ and $Z, Z' \in \mathcal{Z}$, one has $T' \in \mathcal{T}$ and $\chi = \eta_{T'}$.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (M') and (S) , the reflection $\mathbb{F}$ forms an admissible Galois structure.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (M') and (S) , the reflection F forms an admissible Galois structure.

Definition

A functor is $\mathcal{Z}$ -protoadditive if it preserves the pullbacks of split epimorphisms along morphisms inverted by $\mathcal{Z}$.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (M') and (S) , the reflection F forms an admissible Galois structure.

Definition

A functor is $\mathcal{Z}$ -protoadditive if it preserves the pullbacks of split epimorphisms along morphisms inverted by $\mathcal{Z}$.

We say that a torsion theory $(\mathcal{T}, \mathcal{F})$ satisfies (P) if the reflector F is $\mathcal{Z}$ -protoadditive.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (M') , (S) and (P) and $f: C \rightarrow D$ is an effective descent morphism, the following conditions are equivalent:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (M') , (S) and (P) and $f: C \rightarrow D$ is an effective descent morphism, the following conditions are equivalent:

- 1 *f is a normal extension;*

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (M') , (S) and (P) and $f: C \rightarrow D$ is an effective descent morphism, the following conditions are equivalent:

- ① f is a normal extension;*
- ② f is a central extension;*

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory satisfying (M') , (S) and (P) and $f: C \rightarrow D$ is an effective descent morphism, the following conditions are equivalent:

- ❶ *f is a normal extension;*
- ❷ *f is a central extension;*
- ❸ *$K[f] \in \mathcal{F}$.*

In $\text{Arr}(\mathcal{C})$ we consider the class of zeros $\text{Arr}(\mathcal{Z})$.

In $\text{Arr}(\mathcal{C})$ we consider the class of zeros $\text{Arr}(\mathcal{Z})$.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, then $(\mathcal{T}_1, \mathcal{F}_1)$ is a torsion theory in $\text{Arr}(\mathcal{C})$, where

In $\text{Arr}(\mathcal{C})$ we consider the class of zeros $\text{Arr}(\mathcal{Z})$.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, then $(\mathcal{T}_1, \mathcal{F}_1)$ is a torsion theory in $\text{Arr}(\mathcal{C})$, where

$$\mathcal{F}_1 = \{m \in \text{Arr}(\mathcal{C}) \mid K[m] \in \mathcal{F}\},$$

$$\mathcal{T}_1 = \{g \in \text{Arr}(\mathcal{C}) \mid g: T \rightarrow Z, T \in \mathcal{T}, Z \in \mathcal{Z}\}.$$

In $\text{Arr}(\mathcal{C})$ we consider the class of zeros $\text{Arr}(\mathcal{Z})$.

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, then $(\mathcal{T}_1, \mathcal{F}_1)$ is a torsion theory in $\text{Arr}(\mathcal{C})$, where

$$\mathcal{F}_1 = \{m \in \text{Arr}(\mathcal{C}) \mid K[m] \in \mathcal{F}\},$$

$$\mathcal{T}_1 = \{g \in \text{Arr}(\mathcal{C}) \mid g: T \rightarrow Z, T \in \mathcal{T}, Z \in \mathcal{Z}\}.$$

Proposition

If $(\mathcal{T}, \mathcal{F})$ satisfies condition (M) (resp. (S)), then $(\mathcal{T}_1, \mathcal{F}_1)$ satisfies condition (M) (resp. (S)), too.

Condition (N) for the derived torsion theory

Given $f: A \rightarrow B$, $g: A \rightarrow C$ in $\mathcal{C}$, it is not true in general that

$$K[f] \cap K[g] \cong K[\langle f, g \rangle].$$

Condition (N) for the derived torsion theory

Given $f: A \rightarrow B$, $g: A \rightarrow C$ in $\mathcal{C}$, it is not true in general that

$$K[f] \cap K[g] \cong K[\langle f, g \rangle].$$

However:

Condition (N) for the derived torsion theory

Given $f: A \rightarrow B$, $g: A \rightarrow C$ in $\mathcal{C}$, it is not true in general that

$$K[f] \cap K[g] \cong K[\langle f, g \rangle].$$

However:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, then

$$\mathrm{T}(K[f] \cap K[g]) \cong \mathrm{T}(K[\langle f, g \rangle]).$$

Condition (N) for the derived torsion theory

Given $f: A \rightarrow B$, $g: A \rightarrow C$ in $\mathcal{C}$, it is not true in general that

$$K[f] \cap K[g] \cong K[\langle f, g \rangle].$$

However:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, then

$$\mathrm{T}(K[f] \cap K[g]) \cong \mathrm{T}(K[\langle f, g \rangle]).$$

As a consequence:

Condition (N) for the derived torsion theory

Given $f: A \rightarrow B$, $g: A \rightarrow C$ in $\mathcal{C}$, it is not true in general that

$$K[f] \cap K[g] \cong K[\langle f, g \rangle].$$

However:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, then

$$\mathrm{T}(K[f] \cap K[g]) \cong \mathrm{T}(K[\langle f, g \rangle]).$$

As a consequence:

Proposition

If $(\mathcal{T}, \mathcal{F})$ satisfies condition (N), then $(\mathcal{T}_1, \mathcal{F}_1)$ satisfies condition (N), too.

Condition (P) for the derived torsion theory

Proposition

Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between regular protomodular categories with $0 \rightarrow 1$ regular epi. If $GZ = ZG$, then G is protoadditive if and only if it preserves split short Z -exact sequences.

Condition (P) for the derived torsion theory

Proposition

Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between regular protomodular categories with $0 \rightarrow 1$ regular epi. If $GZ = ZG$, then G is protoadditive if and only if it preserves split short Z -exact sequences.

As a consequence:

Condition (P) for the derived torsion theory

Proposition

Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between regular protomodular categories with $0 \rightarrow 1$ regular epi. If $GZ = ZG$, then G is protoadditive if and only if it preserves split short Z -exact sequences.

As a consequence:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, TFAE:

- 1 *the radical $T: \mathcal{C} \rightarrow \mathcal{C}$ is protoadditive;*

Condition (P) for the derived torsion theory

Proposition

Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between regular protomodular categories with $0 \rightarrow 1$ regular epi. If $GZ = ZG$, then G is protoadditive if and only if it preserves split short $\mathcal{Z}$ -exact sequences.

As a consequence:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, TFAE:

- ① the radical $T: \mathcal{C} \rightarrow \mathcal{C}$ is protoadditive;*
- ② the radical $T: \mathcal{C} \rightarrow \mathcal{C}$ preserves split short $\mathcal{Z}$ -exact sequences;*

Condition (P) for the derived torsion theory

Proposition

Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between regular protomodular categories with $0 \rightarrow 1$ regular epi. If $GZ = ZG$, then G is protoadditive if and only if it preserves split short $\mathcal{Z}$ -exact sequences.

As a consequence:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, TFAE:

- 1 the radical $T: \mathcal{C} \rightarrow \mathcal{C}$ is protoadditive;*
- 2 the radical $T: \mathcal{C} \rightarrow \mathcal{C}$ preserves split short $\mathcal{Z}$ -exact sequences;*
- 3 $\mathcal{T}$ is hereditary w.r.t. the class of protosplit monomorphisms.*

Condition (P) for the derived torsion theory

Since:

Condition (P) for the derived torsion theory

Since:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$ satisfying condition (M') , then $(\mathcal{T}, \mathcal{F})$ satisfies condition (P) (i.e. $\mathbb{F}: \mathcal{C} \rightarrow \mathcal{F}$ is protoadditive) if and only if the radical $\mathbb{T}: \mathcal{C} \rightarrow \mathcal{C}$ is protoadditive.

Condition (P) for the derived torsion theory

Since:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$ satisfying condition (M') , then $(\mathcal{T}, \mathcal{F})$ satisfies condition (P) (i.e. $\mathbb{F}: \mathcal{C} \rightarrow \mathcal{F}$ is protoadditive) if and only if the radical $\mathbb{T}: \mathcal{C} \rightarrow \mathcal{C}$ is protoadditive.

and, if $\mathcal{T}$ is hereditary w.r.t. the class of protosplit monomorphisms, so is $\mathcal{T}_1$, we get:

Condition (P) for the derived torsion theory

Since:

Proposition

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$ satisfying condition (M') , then $(\mathcal{T}, \mathcal{F})$ satisfies condition (P) (i.e. $\mathbb{F}: \mathcal{C} \rightarrow \mathcal{F}$ is protoadditive) if and only if the radical $\mathbb{T}: \mathcal{C} \rightarrow \mathcal{C}$ is protoadditive.

and, if $\mathcal{T}$ is hereditary w.r.t. the class of protosplit monomorphisms, so is $\mathcal{T}_1$, we get:

Proposition

If $(\mathcal{T}, \mathcal{F})$ satisfies condition (P) and (M') , then $(\mathcal{T}_1, \mathcal{F}_1)$ satisfies condition (P) , too.

In *Heyt*, $(PD, Boole)$ is a torsion theory, where:

In *Heyt*, $(PD, Boole)$ is a torsion theory, where:

$$\mathcal{Z} = \{1, 2\},$$

In *Heyt*, $(PD, Boole)$ is a torsion theory, where:

$$\mathcal{Z} = \{1, 2\},$$

$$PD = \{ H \in Heyt \mid \forall x \in H \neg x \in \{0, 1\} \},$$

In *Heyt*, $(PD, Boole)$ is a torsion theory, where:

$$\mathcal{Z} = \{1, 2\},$$

$$PD = \{ H \in Heyt \mid \forall x \in H \neg x \in \{0, 1\} \},$$

$F: Heyt \rightarrow Boole$ is defined by

$$F(H) = H_{\neg\neg} = \{ x \in H \mid \neg\neg x = x \},$$

In *Heyt*, $(PD, Boole)$ is a torsion theory, where:

$$\mathcal{Z} = \{1, 2\},$$

$$PD = \{ H \in Heyt \mid \forall x \in H \neg x \in \{0, 1\} \},$$

$F: Heyt \rightarrow Boole$ is defined by

$$F(H) = H \neg \neg = \{ x \in H \mid \neg \neg x = x \},$$

$T: Heyt \rightarrow PD$ is defined by

$$T(H) = \{ x \in H \mid \neg x \in \{0, 1\} \}.$$

Examples

In *Heyt*, $(PD, Boole)$ is a torsion theory, where:

$$\mathcal{Z} = \{1, 2\},$$

$$PD = \{ H \in Heyt \mid \forall x \in H \neg x \in \{0, 1\} \},$$

$F: Heyt \rightarrow Boole$ is defined by

$$F(H) = H \neg \neg = \{ x \in H \mid \neg \neg x = x \},$$

$T: Heyt \rightarrow PD$ is defined by

$$T(H) = \{ x \in H \mid \neg x \in \{0, 1\} \}.$$

The unit η is defined by $\eta_H(x) = \neg \neg x$.

Examples

An *MV*-algebra is an algebra $(A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg$ is involutive,

$$\neg 0 \oplus x = \neg 0, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

Examples

An *MV*-algebra is an algebra $(A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg$ is involutive,

$$\neg 0 \oplus x = \neg 0, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

One can define $x \leq y$ if and only if $\neg x \oplus y = 1 = \neg 0$.

Examples

An *MV*-algebra is an algebra $(A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg$ is involutive,

$$\neg 0 \oplus x = \neg 0, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

One can define $x \leq y$ if and only if $\neg x \oplus y = 1 = \neg 0$.

An ideal of an *MV*-algebra A is a non-empty subset I such that

$$x \in I, y \leq x \Rightarrow y \in I, \quad x, y \in I \Rightarrow x \oplus y \in I.$$

Examples

An *MV*-algebra is an algebra $(A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg$ is involutive,

$$\neg 0 \oplus x = \neg 0, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

One can define $x \leq y$ if and only if $\neg x \oplus y = 1 = \neg 0$.

An ideal of an *MV*-algebra A is a non-empty subset I such that

$$x \in I, y \leq x \Rightarrow y \in I, \quad x, y \in I \Rightarrow x \oplus y \in I.$$

The radical $Rad(A)$ of an *MV*-algebra A is the intersection of the maximal ideals.

Examples

An *MV*-algebra is an algebra $(A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg$ is involutive,

$$\neg 0 \oplus x = \neg 0, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

One can define $x \leq y$ if and only if $\neg x \oplus y = 1 = \neg 0$.

An ideal of an *MV*-algebra A is a non-empty subset I such that

$$x \in I, y \leq x \Rightarrow y \in I, \quad x, y \in I \Rightarrow x \oplus y \in I.$$

The radical $Rad(A)$ of an *MV*-algebra A is the intersection of the maximal ideals. A is semisimple if $Rad(A) = \{0\}$.

Examples

An *MV*-algebra is an algebra $(A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg$ is involutive,

$$\neg 0 \oplus x = \neg 0, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

One can define $x \leq y$ if and only if $\neg x \oplus y = 1 = \neg 0$.

An ideal of an *MV*-algebra A is a non-empty subset I such that

$$x \in I, y \leq x \Rightarrow y \in I, \quad x, y \in I \Rightarrow x \oplus y \in I.$$

The radical $Rad(A)$ of an *MV*-algebra A is the intersection of the maximal ideals. A is semisimple if $Rad(A) = \{0\}$. A is perfect if $A = Rad(A) \cup \neg Rad(A)$.

In MV , (pMV, sMV) is a torsion theory, where:

In MV , (pMV, sMV) is a torsion theory, where:

$$\mathcal{Z} = \{1, 2\},$$

In MV , (pMV, sMV) is a torsion theory, where:

$$\mathcal{Z} = \{1, 2\},$$

$F: MV \rightarrow sMV$ is defined by

$$F(A) = A/Rad(A),$$

In MV , (pMV, sMV) is a torsion theory, where:

$$\mathcal{Z} = \{1, 2\},$$

$F: MV \rightarrow sMV$ is defined by

$$F(A) = A/Rad(A),$$

$T: MV \rightarrow pMV$ is defined by

$$T(A) = P(A) = Rad(A) \cup \neg Rad(A).$$

In $MSet^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

In $MSet^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

In $MSet^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

$$\mathcal{F} = \{X \in MSet \mid Fix(X) = X\},$$

In $MSet^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

$$\mathcal{F} = \{X \in MSet \mid Fix(X) = X\},$$

$F: MSet^{op} \rightarrow \mathcal{F}$ is defined by $F(X) = Fix(X)$,

In $MSet^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

$$\mathcal{F} = \{X \in MSet \mid Fix(X) = X\},$$

$F: MSet^{op} \rightarrow \mathcal{F}$ is defined by $F(X) = Fix(X)$,

$$\mathcal{T} = \{X \in MSet \mid |Fix(X)| \leq 1\},$$

In $MSet^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

$$\mathcal{F} = \{X \in MSet \mid Fix(X) = X\},$$

$F: MSet^{op} \rightarrow \mathcal{F}$ is defined by $F(X) = Fix(X)$,

$$\mathcal{T} = \{X \in MSet \mid |Fix(X)| \leq 1\},$$

$T: MSet^{op} \rightarrow \mathcal{T}$ is defined by $T(X) = X/Fix(X)$.

Examples

A simplicial set is a functor $X: \Delta^{op} \rightarrow \mathbf{Set}$; one writes $X_n = X(n)$.

Examples

A simplicial set is a functor $X: \Delta^{op} \rightarrow \mathbf{Set}$; one writes $X_n = X(n)$.
In $s\mathbf{Set}^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

Examples

A simplicial set is a functor $X: \Delta^{op} \rightarrow \mathbf{Set}$; one writes $X_n = X(n)$.
In $s\mathbf{Set}^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

Examples

A simplicial set is a functor $X: \Delta^{op} \rightarrow \mathbf{Set}$; one writes $X_n = X(n)$.
In $s\mathbf{Set}^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

$$\mathcal{F} = \{X \in s\mathbf{Set} \mid X_n = X_0, X(f) = id_{X_0} \text{ for all } n \in \mathbb{N}, f \in \mathbf{Arr}(\Delta)\},$$

Examples

A simplicial set is a functor $X: \Delta^{op} \rightarrow \mathbf{Set}$; one writes $X_n = X(n)$.
In $s\mathbf{Set}^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

$$\mathcal{F} = \{X \in s\mathbf{Set} \mid X_n = X_0, X(f) = id_{X_0} \text{ for all } n \in \mathbb{N}, f \in \mathbf{Arr}(\Delta)\},$$

$F: s\mathbf{Set}^{op} \rightarrow \mathcal{F}$ is defined by $F(X)_n = X_0$ for all n ,

Examples

A simplicial set is a functor $X: \Delta^{op} \rightarrow \mathbf{Set}$; one writes $X_n = X(n)$.
In $s\mathbf{Set}^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

$$\mathcal{F} = \{X \in s\mathbf{Set} \mid X_n = X_0, X(f) = id_{X_0} \text{ for all } n \in \mathbb{N}, f \in \mathbf{Arr}(\Delta)\},$$

$F: s\mathbf{Set}^{op} \rightarrow \mathcal{F}$ is defined by $F(X)_n = X_0$ for all n ,

$$\mathcal{T} = \{X \in s\mathbf{Set} \mid |X_0| \leq 1\},$$

Examples

A simplicial set is a functor $X: \Delta^{op} \rightarrow \mathbf{Set}$; one writes $X_n = X(n)$.
In $s\mathbf{Set}^{op}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{Z} = \{1, \emptyset\},$$

$$\mathcal{F} = \{X \in s\mathbf{Set} \mid X_n = X_0, X(f) = id_{X_0} \text{ for all } n \in \mathbb{N}, f \in \mathbf{Arr}(\Delta)\},$$

$F: s\mathbf{Set}^{op} \rightarrow \mathcal{F}$ is defined by $F(X)_n = X_0$ for all n ,

$$\mathcal{T} = \{X \in s\mathbf{Set} \mid |X_0| \leq 1\},$$

$T: s\mathbf{Set}^{op} \rightarrow \mathcal{T}$ is defined, if $X \neq \emptyset$, by the pushout

$$\begin{array}{ccc} F(X) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & T(X). \end{array}$$

If $m > 0$, an abelian group A is m -divisible if for all $a \in A$ there exists $b \in A$ such that $a = mb$.

Examples

If $m > 0$, an abelian group A is m -divisible if for all $a \in A$ there exists $b \in A$ such that $a = mb$.

Examples: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$,

Examples

If $m > 0$, an abelian group A is m -divisible if for all $a \in A$ there exists $b \in A$ such that $a = mb$.

Examples: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\{z \in \mathbb{C} \mid \exists n \in \mathbb{N} : z^{m^n} = 1\}$.

Examples

If $m > 0$, an abelian group A is m -divisible if for all $a \in A$ there exists $b \in A$ such that $a = mb$.

Examples: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\{z \in \mathbb{C} \mid \exists n \in \mathbb{N} : z^{m^n} = 1\}$.

We define $D_m(A) = \{a \in A \mid \exists (a_n) : a_0 = a, a_n = ma_{n+1}\}$.

If $m > 0$, an abelian group A is m -divisible if for all $a \in A$ there exists $b \in A$ such that $a = mb$.

Examples: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\{z \in \mathbb{C} \mid \exists n \in \mathbb{N} : z^{m^n} = 1\}$.

We define $D_m(A) = \{a \in A \mid \exists (a_n) : a_0 = a, a_n = ma_{n+1}\}$.

We consider the category $\mathbb{Z}_m/Ab$.

If $m > 0$, an abelian group A is m -divisible if for all $a \in A$ there exists $b \in A$ such that $a = mb$.

Examples: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\{z \in \mathbb{C} \mid \exists n \in \mathbb{N} : z^{m^n} = 1\}$.

We define $D_m(A) = \{a \in A \mid \exists (a_n) : a_0 = a, a_n = ma_{n+1}\}$.

We consider the category $\mathbb{Z}_m/Ab$. Its objects can be seen as pairs (A, a) where A is an abelian group and $a \in A$ is such that $ma = 0$.

Examples

If $m > 0$, an abelian group A is m -divisible if for all $a \in A$ there exists $b \in A$ such that $a = mb$.

Examples: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\{z \in \mathbb{C} \mid \exists n \in \mathbb{N} : z^{m^n} = 1\}$.

We define $D_m(A) = \{a \in A \mid \exists (a_n) : a_0 = a, a_n = ma_{n+1}\}$.

We consider the category $\mathbb{Z}_m/Ab$. Its objects can be seen as pairs (A, a) where A is an abelian group and $a \in A$ is such that $ma = 0$.

Let $\mathcal{Z}$ be the subcategory of pairs $(\mathbb{Z}_h, 1)$ where h divides m .

In $\mathbb{Z}_m/Ab$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

In $\mathbb{Z}_m/\text{Ab}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{F} = \{ (A, a) \mid D_m(A) = 0 \},$$

In $\mathbb{Z}_m/Ab$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{F} = \{ (A, a) \mid D_m(A) = 0 \},$$

$F: \mathbb{Z}_m/Ab \rightarrow \mathcal{F}$ is defined by $F(A, a) = (A/D_m(A), [a])$,

In $\mathbb{Z}_m/Ab$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{F} = \{ (A, a) \mid D_m(A) = 0 \},$$

$F: \mathbb{Z}_m/Ab \rightarrow \mathcal{F}$ is defined by $F(A, a) = (A/D_m(A), [a])$,

$$\mathcal{T} = \{ (A, a) \mid A = \langle D_m(A), a \rangle \},$$

In $\mathbb{Z}_m/\text{Ab}$, $(\mathcal{T}, \mathcal{F})$ is a torsion theory, where:

$$\mathcal{F} = \{ (A, a) \mid D_m(A) = 0 \},$$

$F: \mathbb{Z}_m/\text{Ab} \rightarrow \mathcal{F}$ is defined by $F(A, a) = (A/D_m(A), [a])$,

$$\mathcal{T} = \{ (A, a) \mid A = \langle D_m(A), a \rangle \},$$

$T: \mathbb{Z}_m/\text{Ab} \rightarrow \mathcal{T}$ is defined by $T(A, a) = (\langle D_m(A), a \rangle, a)$.