

Affine spaces, naturally Mal'cev categories, and descent

PSSL113

UCLouvain – 26 September 2026

Mariano Messori (IUSS Pavia)
Joint work with Sandra Mantovani

Affine spaces

Given a commutative unital ring k (typically a field) a k -affine space is given by

- a (non-empty) set A
- a free transitive action of a k -module M on A

A (non-empty) k -affine space can equivalently be described as a surjective k -linear map with codomain k :

- ➡ Given a surjective k -linear map $\rho: W \rightarrow k$, then $\rho^{-1}(0) = \ker \rho$ acts freely and transitively on the non-empty set $\rho^{-1}(1)$
- ➡ Every (non-empty) affine space arises this way (up to iso)

Affine spaces and slices

Thus, affine spaces on a ring k can be described as morphisms in an additive category $(= \mathbf{Mod}_k)$ with a fixed codomain $(= k)$

➡ It is interesting to study slices of additive categories

- Slices of (finitely complete) additive categories were characterised by A. Carboni (1989) in terms of algebraic properties ➡ “modular categories”
- Another characterisation was developed by A. Carboni and G. Janelidze (1995) in terms of descent properties

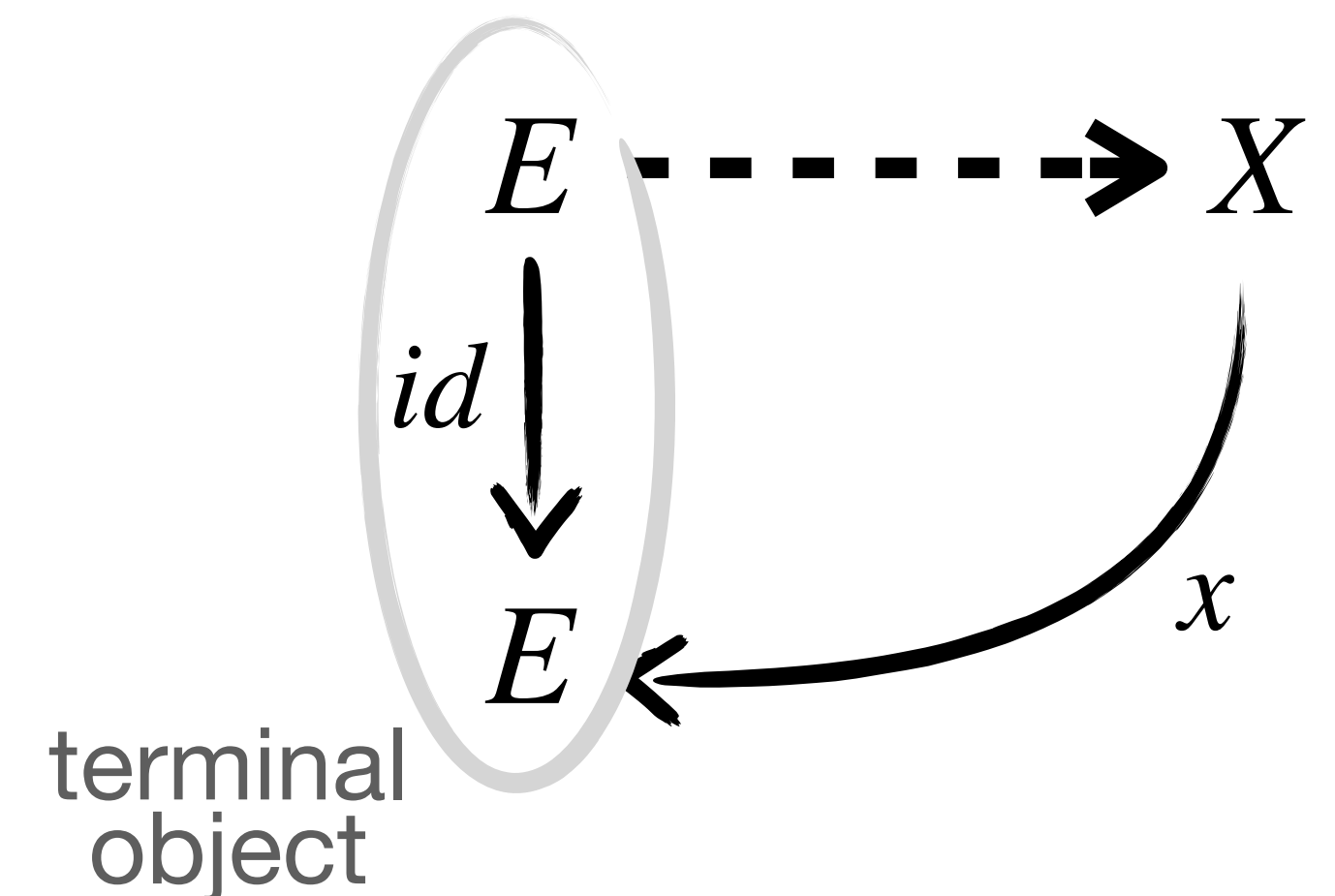
Affine spaces and slices

- The slice $\mathbf{Mod}_k/k$ contains all maps with codomain k , while non-empty affine spaces correspond to just surjective maps
 - ➡ non-empty affine spaces form a full subcategory of $\mathbf{Mod}_k/k$
- Adding non-surjective maps raises several issues:
 - ▶ introduces infinitely many empty affine spaces $(\emptyset, M)$, one for each null map $M \xrightarrow{0} k$
 - ▶ when k is not a field, there can be non-isomorphic maps in the slice corresponding to the same (empty) affine space
 - ▶ these empty affine spaces cannot just be discarded, as they appear as limits and colimits of non-empty ones

Subcategories of slices of additive categories

- To avoid these issues we will focus on special subcategories of slices of additive categories. In particular, we consider those full subcategories which:
 - admit suitable limits and colimits;
 - “contain non-empty affine spaces” = contain all split epis with the prescribed codomain
- An affine space A is non-empty iff it admits a map $1 \rightarrow A$ from the terminal object

$\Rightarrow$ In a slice $\mathcal{X}/E$: terminal object = $(E \xrightarrow{id} E)$
 $\implies (X \xrightarrow{x} E)$ admits a map from the terminal object
 iff x is a split epimorphism



Theorem. Let $\mathcal{A}$ be a category with finite limits and finite coproducts. The following are equivalent.

- (1) There exists an additive finitely complete category $\mathcal{X}$ and an object $E \in \mathcal{X}$ such that $\mathcal{A}$ is equivalent to a full subcategory of the slice $\mathcal{X}/E$ containing all split epis of codomain E .
- (2) The following conditions hold:
 - (a) $1/\mathcal{A}$ is additive;
 - (b) $0 \rightarrow 1$ is a codescent morphism in $\mathcal{A}$.

In this case $1/\mathcal{A} \simeq \mathcal{X}$.

Compare:

Theorem (Carboni, Janelidze, 1995).

Let $\mathcal{A}$ be a category with finite limits and finite coproducts. The following are equivalent.

- (1) $\mathcal{A}$ is equivalent to a slice of an additive category
- (2) The following conditions hold:
 - (a) $1/\mathcal{A}$ is additive
 - (b) $0 \rightarrow 1$ is an effective codescent morphism in $\mathcal{A}$

Examples

For a commutative unital ring k , let $\mathcal{A}$ be the full subcategory of $\mathbf{Mod}_k/k$ consisting of maps $\varphi: M \rightarrow k$ such that φ is surjective or $M = 0$

- $\implies$
- $\mathcal{A}$ satisfies the conditions of the theorem
 - $\mathcal{A}$ is complete, cocomplete, Barr exact, and coregular
 - $\mathcal{A}$ is equivalent to the category of all non-empty affine spaces + a single “empty affine space” which is strictly initial (even for k not a field)

Similarly: for any additive category $\mathcal{X} \implies$ subcategories of $\mathcal{X}/E$ determined by epis, split epis...

Theorem. Let $\mathcal{A}$ be a category with finite limits and finite coproducts. The following are equivalent.

- (1) There exists an additive finitely complete category $\mathcal{X}$ and an object $E \in \mathcal{X}$ such that $\mathcal{A}$ is equivalent to a full subcategory of the slice $\mathcal{X}/E$ containing all split epis of codomain E .
- (2) The following conditions hold:
 - (a) $0 \rightarrow 1$ is a codescent morphism in $\mathcal{A}$
 - (b) $1/\mathcal{A}$ is additive
- (3) The following conditions hold:
 - (a) $0 \rightarrow 1$ is a codescent morphism in $\mathcal{A}$
 - (b') $\mathcal{A}$ is naturally Mal'cev

- A **Mal'cev operation** on an object X is a ternary operation $\mu: X^3 \rightarrow X$ such that $\mu(x, x, y) = y$ and $\mu(x, y, y) = x$
- **Example:** for a group $(A, +)$, $\mu(x, y, z) = x - y + z$
- A **naturally Mal'cev category** (Johnstone, 1989) is a category where every object admits a Mal'cev operation preserved by all morphisms
- Naturally Mal'cev + pointed = additive

Theorem. Let $\mathcal{A}$ be a category with finite limits and finite coproducts. The following are equivalent.

- (1) There exists an additive finitely complete category $\mathcal{X}$ and an object $E \in \mathcal{X}$ such that $\mathcal{A}$ is equivalent to a full subcategory of the slice $\mathcal{X}/E$ containing all split epis of codomain E .
- (2) The following conditions hold:
 - (a) $0 \rightarrow 1$ is a codescent morphism in $\mathcal{A}$
 - (b) $1/\mathcal{A}$ is additive
- (3) The following conditions hold:
 - (a) $0 \rightarrow 1$ is a codescent morphism in $\mathcal{A}$
 - (b') $\mathcal{A}$ is naturally Mal'cev

“Varietal” description of affine spaces

an affine space on k can be defined as a set with a ternary operation μ_t for each $t \in k$ + some axioms

to be thought of as

$$“\mu_t(A, B, C) = t(A - B) + C”$$

➡ for $t = 1$, μ_1 is a natural Mal'cev operation

➡ This variety is equivalent to the subcategory of $\mathbf{Mod}_k/k$ of the previous slide

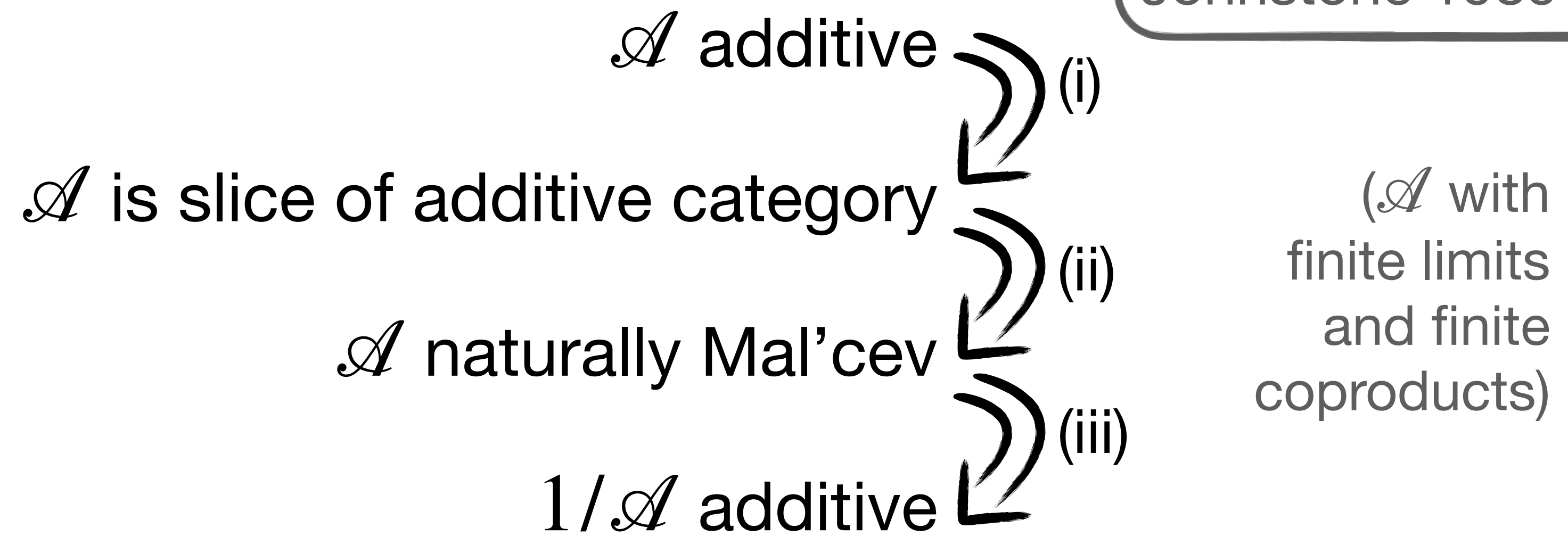
Theorem. Let $\mathcal{A}$ be a category with finite limits and finite coproducts. The following are equivalent.

Assume $0 \rightarrow 1$ is a codescent morphism

The following are equivalent.

(b) $1/\mathcal{A}$ is additive

(b') $\mathcal{A}$ is naturally Mal'cev



- The implications cannot be reversed in general
- When $0 \rightarrow 1$ iso: all three implications can be reversed
- When $0 \rightarrow 1$ effective codescent: (ii) and (iii) can be reversed
- When $0 \rightarrow 1$ codescent: (iii) can be reversed

Carboni,
Janelidze 1995

References

- A. Carboni. 'Categories of affine spaces'. *Journal of Pure and Applied Algebra* 61.3 (1989).
- A. Carboni, G. Janelidze. 'Modularity and descent'. *Journal of Pure and Applied Algebra* 99.3 (1995).
- P. T. Johnstone. 'Affine categories and naturally Mal'cev categories'. *Journal of Pure and Applied Algebra* 61.3 (1989).

Thank you for the attention!