

# A Characterization of Abelianness via Lawvere Doctrines

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## Some preliminaries on doctrines

### Definition

A (Lawvere) **doctrine** on a category  $\mathcal{C}$  with finite products is a functor  $P : \mathcal{C}^{op} \rightarrow Pos$ , where  $Pos$  is a suitable category of posets (e.g. meet-semilattices, bounded lattices...).

Some classes of doctrines:

- **Primary:** Fibres  $P(A)$  are meet-semilattices,  $P(f) =: P_f$  preserve meets;
- **Elementary:** Fibres  $P(A \times A)$  possess an equality predicate  $\delta_A$ ;
- **Existential:** Every  $P_f$  has a left adjoint  $\exists_f$ , and these satisfy the Beck-Chevalley (BC) and the Frobenius (FR) conditions.

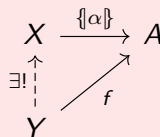
Elementary, existential doctrines are able to express the fragment  $(\exists, \top, =, \wedge)$  of logic.

# Comprehensions

## Definition

Let  $P : \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$  be a primary doctrine. A strong **comprehension** of an element  $\alpha \in P(A)$  is a morphism  $\{\alpha\} : X \rightarrow A$  such that  $P_{\{\alpha\}}(\alpha) = \top_X$  and every  $f : Y \rightarrow A$  satisfying  $P_f(\alpha) = \top_Y$  factors uniquely through  $\{\alpha\}$ .

Comprehensions are **full** when  $\alpha \leq \beta \Leftrightarrow \{\alpha\}$  factors through  $\{\beta\}$  holds for every  $\alpha, \beta$ .



Fullness ensures that the **posetal structure** of the fibre of  $A$  is brought over to  $\mathbf{Sub}(A)$ .

# The example of regular categories

Consider the following:

## Theorem

*Let  $P : \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$  be an elementary, existential doctrine. The following are equivalent:*

- $\mathcal{C}$  is **regular** and  $P \cong \mathbf{Sub}_{\mathcal{C}}$ ;
- $P$  has full, strong comprehensions, extensional equality and satisfies the Rule of Unique Choice.

## Remark

Analogous characterizations exist for toposes, coherent categories, Heyting categories...

B. Jacobs. *Categorical logic and type theory*. Elsevier Science, 1999.

M.E. Maietti, F. Pasquali, and G. Rosolini. "Triplices, exact completions, and Hilbert's  $\epsilon$ -operator". *Tbilisi Mathematical Journal* 10.3 (2017), pp. 141–166.

# Main aim of the talk

- Can we obtain a similar characterization for **abelian** categories?
- What about **semi-abelian** categories?

Such characterizations were obtained in terms of **forms**: thus, we also wish to investigate the **connection** between forms and doctrines.

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Z. Janelidze and F. van Niekerk. “Toposes have an optimal noetherian form”. *Journal of Pure and Applied Algebra* 230.10 (2026), pp. 108–340.

# Orean doctrines

## Definition

Let  $\mathcal{C}$  be a category with finite products and coproducts. An **orean** doctrine is a primary doctrine  $P : \mathcal{C}^{op} \rightarrow Pos$  such that:

- The fibres of  $P$  are **bounded lattices**  $(P(A), \wedge, \vee, \perp, \top)$ ;
- $P$  has all left adjoints  $\exists_f \dashv P_f$ .

**Orean** doctrines provide the **minimal** structure necessary to define:

- **$P$ -kernels** as elements of the form  $P_f(\perp)$  for some  $f$ ;
- **$P$ -images** as elements of the form  $\exists_f(\top)$  for some  $f$ .

We also need to abstract the notions of kernel and cokernel morphisms, by dualizing the notion of comprehension introduced previously:

### Definition

Let  $P$  be an orean doctrine. A strong **comprehension** of  $\alpha \in P(A)$  is a morphism  $\{\!\!\{\alpha}\!\!\} : X \rightarrow A$  such that:

- $P_{\{\!\!\{\alpha}\!\!\}}(\alpha) = \top$ ;
- Every  $f : Y \rightarrow A$  such that  $P_f(\alpha) = \top$  factors uniquely through  $\{\!\!\{\alpha}\!\!\}$ .

Dually, a strong **co-comprehension** of  $\alpha \in P(A)$  is a morphism  $\overline{\{\!\!\{\alpha}\!\!\}} : A \rightarrow X'$  such that:

- $\exists_{\overline{\{\!\!\{\alpha}\!\!\}}}(\alpha) = \perp$ ;
- Every  $g : Y' \rightarrow X'$  such that  $\exists_g(\alpha) = \perp$  factors uniquely through  $\overline{\{\!\!\{\alpha}\!\!\}}$ .

## An example of orean doctrine

An important class of examples is that of **subobject** doctrines on categories of group-like structures: consider, for example,  $Sub_G$ . In this case:

- The subgroups of a fixed group  $G$  form a **bounded lattice**, where  $A \wedge B = A \cap B$  and  $A \vee B = \langle A, B \rangle$ ;
- Given a group homomorphism  $f : G \rightarrow H$ ,  $\exists_f \dashv P_f$  are, respectively, the **direct** and **inverse** images along  $f$ ;
- If  $S \subseteq G$  is a subgroup, its **comprehension** is the inclusion  $S \hookrightarrow G$ ;
- Only **normal** subgroups  $N$  of  $G$  admit a **co-comprehension**, which is the quotient projection  $G \twoheadrightarrow G/N$ .

We obtained the following result:

### Theorem

A category  $\mathcal{C}$  with finite products and coproducts is **abelian** if and only if it is equipped with an orean doctrine  $P : \mathcal{C}^{op} \rightarrow Pos$  satisfying:

- (i)  $P$  admits full, strong **comprehensions** satisfying (BC);
- (ii)  $P$  admits full, strong **co-comprehensions** satisfying (BC);
- (iii) We have

$$\top \leq \exists_{\overline{\{\alpha\}}}(\top), \quad P_{\{\alpha\}}(\perp) \leq \perp$$

for every  $\alpha$  in  $P(A)$ .

## Theorem

A **pointed** category  $\mathcal{C}$  with finite products and coproducts is **semi-abelian** if and only if it is equipped with an orean doctrine  $P : \mathcal{C}^{op} \rightarrow Pos$  satisfying:

- (i)  $P$  admits full, strong **comprehensions** satisfying (BC);
- (ii') Every  $P$ -**kernel** admits a strong **co-embedding** (i.e. an universal arrow such that  $P_{\overline{\{\alpha\}}}(\perp) = \alpha$ ), and these are composable;
- (iii) We have

$$\top \leq \exists_{\overline{\{\alpha\}}}(\top), \quad P_{\{\alpha\}}(\perp) \leq \perp$$

for every  $\alpha$  in  $P(A)$ .

Moreover, we can rewrite the results in terms of an existential doctrine with comprehensions  $P$  and of its **dual**  $\overline{P}$ ; these must satisfy the compatibility condition (iii).

# Factorization systems induced by doctrines

## Theorem

Let  $\mathcal{C}$  be a category admitting pullbacks. There is a bijection between **proper, stable factorization systems** on  $\mathcal{C}$  and isomorphism classes of **elementary, existential doctrines**  $P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL}$  admitting full, strong comprehensions satisfying (BC) and having extensional equality.

If  $P$  is such a doctrine, every morphism of  $\mathcal{C}$  factors as

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 & \searrow e & \nearrow \{\exists_f(\top)\} \\
 & \bullet &
 \end{array}$$

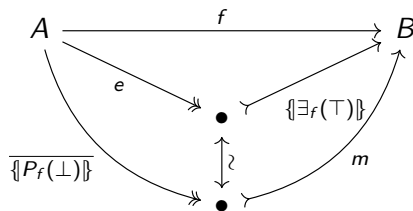
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J. Hughes and B. Jacobs. "Factorization systems and fibrations: Toward a fibred Birkhoff variety theorem". *Electronic Notes in Theoretical Computer Science* 69 (2003), pp. 156–182.

An existential doctrine satisfying (i) therefore induces such a factorization system  $(\mathcal{E}_P, \mathcal{M}_P)$ ; so does its dual  $\bar{P} : \mathcal{C} \rightarrow \text{Pos}$ , if it satisfies (ii). We must then require that the two **coincide**, i.e.

$$(\mathcal{E}_P, \mathcal{M}_P) = (\mathcal{E}_{\bar{P}}, \mathcal{M}_{\bar{P}}).$$

In this way we obtain the (normal epi, normal mono)-factorization system on an abelian category  $\mathcal{C}$ .



# Forms

## Definition

A **form** over a category  $\mathcal{C}$  is a faithful, amnestic functor  $F : \mathcal{B} \rightarrow \mathcal{C}$ .

Forms were introduced as a self-dual framework for recovering homomorphism theorems in non-abelian categories.

There is a bijection between the classes of orean doctrines and orean forms (i.e. Grothendieck bifibrations whose fibres are bounded lattices).

$$(F : \mathcal{B} \longrightarrow \mathcal{C}) \Rightarrow (F^{-1} : \mathcal{C}^{op} \longrightarrow Pos)$$

$$(\int P : \mathcal{G}_P \longrightarrow \mathcal{C}) \Leftarrow (P : \mathcal{C}^{op} \longrightarrow Pos)$$

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Z. Janelidze and T. Weighill. “Duality in non-abelian algebra II. From Isbell bicategories to Grandis exact categories”. *Journal of Homotopy and Related Structures* 11.3 (2016), pp. 553–570.

Our proof relies on the following result:

### Theorem

*Let  $\mathcal{C}$  be a category with finite products and coproducts. Then:*

- $\mathcal{C}$  is **abelian** if and only if it admits a **binormal noetherian form**;
- $\mathcal{C}$  is **semi-abelian** if and only if it is **pointed** and it admits a **conormal noetherian form**.

### Remark

**Conormal** elements are those of the form  $\exists_f(\top)$ ; **normal** elements are those of the form  $P_g(\perp)$ .

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## Relationship with categorical duality

Noetherian forms possess all left adjoints  $\exists_f \dashv P_f$  and satisfy further **self-dual** requirements, such as:

- All adjoints  $\exists_f \dashv P_f$  satisfy a weaker version of the Frobenius condition and its dual;
- Every morphism  $f$  factors as  $f = \{\exists_f(\top)\} \circ \overline{\{P_f(\perp)\}}$ .

On the other hand, the approach via doctrines **unpacks** duality, exploring the connection between (the conormal part of) the axioms of noetherianity and the corresponding notions in the general theory of doctrines.

## A final overview

In the end, we have:

### Theorem

A pointed category  $\mathcal{C}$  with finite products and coproducts is **semi-abelian** if and only if it is equipped with:

- An orean doctrine  $P : \mathcal{C}^{\text{op}} \rightarrow \text{Pos}$  satisfying (i), (ii'), (iii);
- Equivalently, a conormal noetherian form  $F$ .

Moreover,  $\mathcal{C}$  is **abelian** if and only if:

- $P$  also admits full, strong co-comprehensions satisfying (BC);
- Equivalently,  $F$  is binormal.

## Possible future developments

The **exact completion** of an elementary doctrine  $P$  is **linked** to the **ex/wlex completion** of a weakly lex category - for example, they coincide when  $P$  is obtained as a pure existential completion of a subdoctrine  $P'$ .

Weakly lex categories whose **ex/wlex completion** is (semi-)abelian have been described in the works cited below.

Under which hypotheses on a doctrine  $P$  does one have that its exact completion  $Ex(P)$  is a **(semi-)abelian** category?

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M. Gran. "Semi-abelian exact completions". *Homology, Homotopy and Applications* 4.1 (2002), pp. 175–189.

A. Carboni. "Some free constructions in realizability and proof theory". *Journal of Pure and Applied Algebra* 103.2 (1995), pp. 117–148.

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- [2] M. Gran. "Semi-abelian exact completions". *Homology, Homotopy and Applications* 4.1 (2002), pp. 175–189.
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- [5] Z. Janelidze and F. van Niekerk. "Toposes have an optimal noetherian form". *Journal of Pure and Applied Algebra* 230.10 (2026), pp. 108–340.
- [6] Z. Janelidze and T. Weighill. "Duality in non-abelian algebra II. From Isbell bicategories to Grandis exact categories". *Journal of Homotopy and Related Structures* 11.3 (2016), pp. 553–570.
- [7] M.E. Maietti, F. Pasquali, and G. Rosolini. "Tripeses, exact completions, and Hilbert's  $\epsilon$ -operator". *Tbilisi Mathematical Journal* 10.3 (2017), pp. 141–166.

Thank you for your attention!