

# Normalizers in Heyting semilattices

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# Definition of a Heyting semilattice

A Heyting semilattice is an algebra  $\mathbf{X} = (X, \wedge, \rightarrow, 1)$  such that

①  $(X, \wedge, 1)$  is a meet-semilattice with largest element 1:

①  $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

②  $x \wedge y = y \wedge x$

③  $x \wedge x = x$

④  $x \wedge 1 = x$

② For all  $x, y, z \in X$

$$z \leq x \rightarrow y \iff x \wedge z \leq y$$

or equivalently:

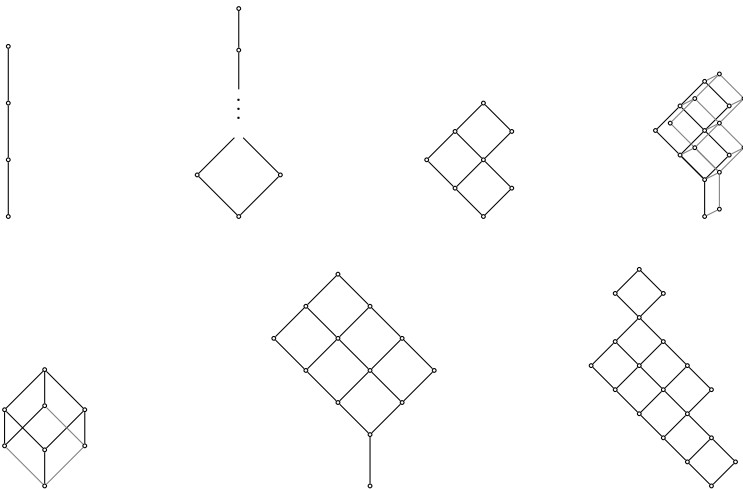
①  $x \rightarrow x = 1$

②  $x \wedge (x \rightarrow y) = x \wedge y$

③  $y \wedge (x \rightarrow y) = y$

④  $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$

# Examples of Heyting semilattices



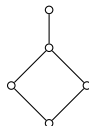
# Equationally linear Heyting semilattices

**ELHSL** (equationally linear Heyting semilattices) is the variety of HSL satisfying the term equation

$$((x \rightarrow y) \rightarrow z) \wedge ((y \rightarrow x) \rightarrow z) = z.$$

It can equivalently also be defined as:

- the variety of Heyting semilattices generated by finite chains
- the class of Heyting semilattices not containing the following Heyting semilattice as a subalgebra:



# Subvarieties of ELHSL

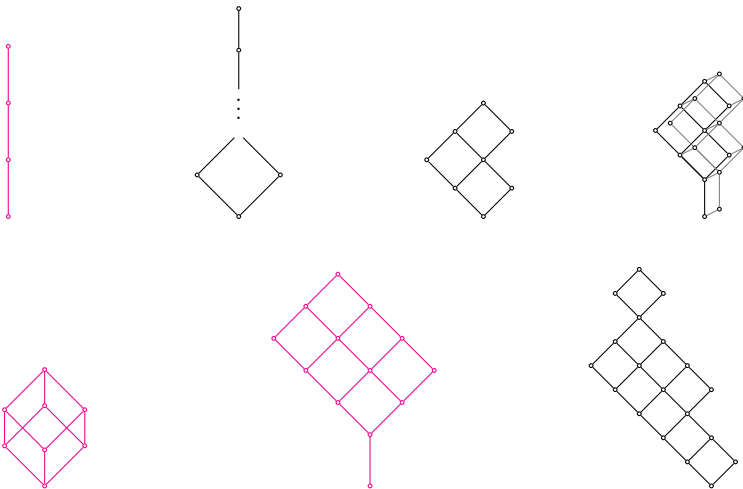
For  $n \in \mathbb{N}$ ,  $\text{ELHSL}_n$  is defined as the subvariety of ELHSL satisfying the additional term equation

$$\bigvee_{i=1}^n (x_i \rightarrow x_{i+1}) = 1.$$

It can equivalently be defined as the subvariety of ELHSL generated by the chain of length  $n$  (see Theorem 3.4 in Nemitz and Whaley [8]).

For  $n \in \mathbb{Z}^+$ , we will denote  $\mathbf{C}_n$  for the chain of length  $n$ .

# Examples of equationally linear Heyting semilattices



In a pointed variety, a subalgebra  $N$  of an algebra  $X$  is said to be normal if  $N = \ker(f)$  for some homomorphism  $f : X \rightarrow Y$ .

The following result is well-known regarding Heyting semilattices, and can be seen in Nemitz [7].

### Lemma

*A subalgebra  $N$  of a Heyting semilattice  $X$  is normal if and only if it is a filter of  $X$  (upclosed and closed under meets).*

# Bergman's Double Projection Theorem

## Theorem (Bergman's Double Projection Theorem for Heyting semilattices)

*Let  $n \in \mathbb{N}$  and consider Heyting semilattices  $X_1, \dots, X_n$ . Let  $A_1, A_2$  be subalgebras of  $X_1 \times X_2 \times \dots \times X_n$ . Then  $A_1 = A_2$  if and only if  $\pi_{ij}(A_1) = \pi_{ij}(A_2)$  for all  $i, j \in \{1, 2, \dots, n\}$  where  $\pi_{ij}$  is the projection onto  $X_i \times X_j$ .*

If  $X \in \text{ELHSL}$  and  $X$  is finite, then  $X$  is a subalgebra of a product  $\mathbf{C}_{k_1} \times \dots \times \mathbf{C}_{k_n}$  (using  $X$ 's subdirect representation).

Hence, if  $A_1, A_2$  are subalgebras of  $X$  then  $A_1 = A_2$  if and only if  $\pi_{ij}(A_1) = \pi_{ij}(A_2)$  for all  $i, j \in \{1, 2, \dots, n\}$  where  $\pi_{ij}$  is the projection onto  $\mathbf{C}_{k_i} \times \mathbf{C}_{k_j}$ .

# Normalizers

The notion of a normalizer is well known in a group: given a subgroup  $H$  of a group  $G$ , the normalizer of  $H$  in  $G$  is defined as the largest subgroup  $N_G(H)$  of  $G$  such that  $H$  is normal in  $N_G(H)$ , and is given by the formula

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}.$$

The notion of a normalizer has been defined in any pointed category (see Gray [5]), and in particular, a normalizer is defined as follows in a pointed variety:

## Definition

Let  $\mathbb{V}$  be a pointed variety and consider an algebra  $X \in \mathbb{V}$ . The **normalizer** of a subalgebra  $S$  of  $X$  (if it exists) is defined as the largest subalgebra  $N_X(S)$  of  $X$  such that  $S$  is normal in  $N_X(S)$ .

A variety  $\mathbb{V}$  *has normalizers* if  $N_X(S)$  exists for all subalgebras  $S$  of algebras  $X \in \mathbb{V}$ .

## Theorem

$\text{ELHSL}_3$  is the largest subvariety of HSL that has normalizers.

*Proof.*

- ① Show  $\text{ELHSL}_3$  has normalizers:
  - Verify  $\text{ELHSL}_3$  has normality of unions
- ② Show that if a subvariety  $\mathbb{V}$  of HSL has normalizers, then  $\mathbb{V} \subseteq \text{ELHSL}_3$ .

In a pointed variety, the existence of normalizers is equivalent to normality of unions (see Gray [5]).

## Theorem

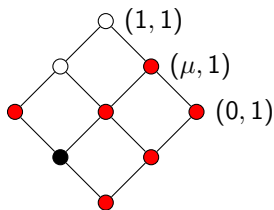
*Let  $C \in \text{ELHSL}_3$  and let  $A$  and  $B$  be subalgebras of  $C$  such that  $A \vee B = C$ . Let  $S$  be a normal subalgebra in  $A$  and  $B$ . Then  $S$  is a normal subalgebra in  $C$ .*

# Proof for normalizers in $\text{ELHSL}_3$

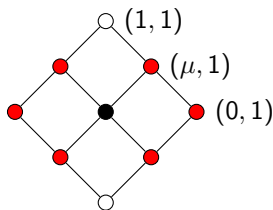
## *Proof outline:*

- Consider when  $C$  is finite
- Let  $s = \bigwedge S$
- We want to show that  $S = \uparrow s$  so that  $S$  is a filter
- Trivially  $S \subseteq \uparrow s$
- Use Bergman's Double Projection Theorem:
- Show  $\pi_i(\uparrow s) \subseteq \pi_i(S)$
- Show  $\pi_{ij}(\uparrow s) \subseteq \pi_{ij}(S)$ : Consider  $(x_i, x_j) \in \pi_{ij}(\uparrow s)$ 
  - **Case 1:**  $x_j = 1$
  - **Case 2:**  $x_j \neq 1$
- Hence,  $S = \uparrow s$  so that normality of unions hold when  $C$  is finite
- Extend to the case when  $C$  is not finite (here we use the fact that  $\text{HSL}$  is locally finite)

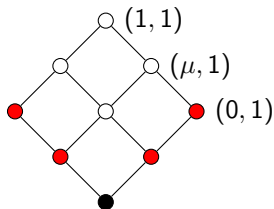
For **case 1**, we assume for contradiction that  $(x_i, 1) \notin \pi_{ij}(S)$ . We can show that there will exist a maximal element  $x \in A$  with  $\pi_{ij}(x) = x_i$ . This will lead to us deducing that  $\pi_{ij}(A)$  will have to exclude the following elements indicated in **red**:



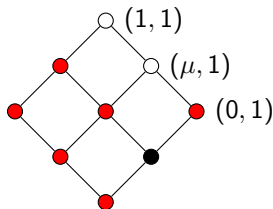
(1)  $\pi_{ij}(x) = (\mu, 0)$



(2)  $\pi_{ij}(x) = (\mu, \mu)$

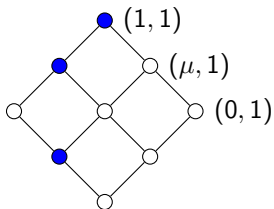


(3)  $\pi_{ij}(x) = (0, 0)$

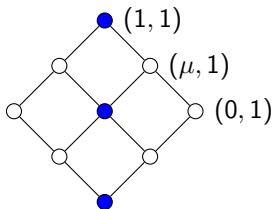


(4)  $\pi_{ij}(x) = (0, \mu)$

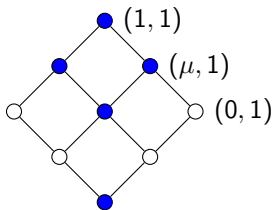
So that  $\pi_{ij}(A)$  will be contained in the following subalgebras of  $\mathbf{C}_3 \times \mathbf{C}_3$  indicated in blue.



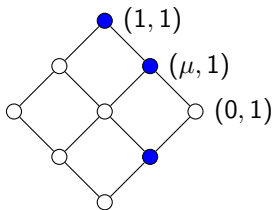
(1)  $\pi_{ij}(x) = (\mu, 0)$



(2)  $\pi_{ij}(x) = (\mu, \mu)$



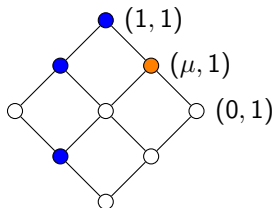
(3)  $\pi_{ij}(x) = (0, 0)$



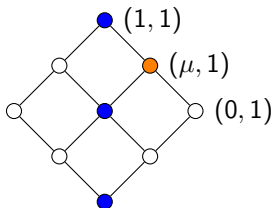
(4)  $\pi_{ij}(x) = (0, \mu)$

Similarly so for  $\pi_{ij}(B)$ . Since  $A \vee B = C$ , we have

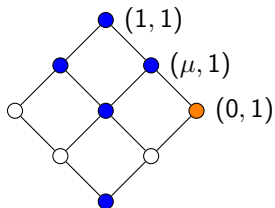
$(x_i, 1) \in \pi_{ij}(C) = \pi_{ij}(A \vee B) = \pi_{ij}(A) \vee \pi_{ij}(B)$  which should be contained in one of the blue elements. However, this does not happen in any of the examples.



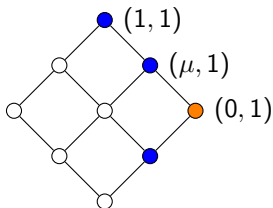
$$(1) \pi_{ij}(x) = (\mu, 0)$$



$$(2) \pi_{ij}(x) = (\mu, \mu)$$



$$(3) \pi_{ij}(x) = (0, 0)$$



$$(4) \pi_{ij}(x) = (0, \mu)$$

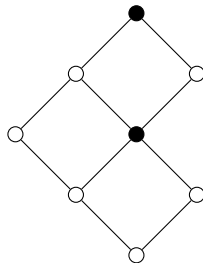
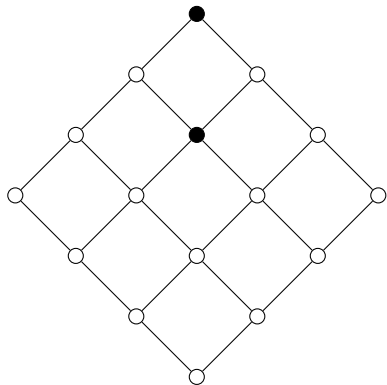
Why is  $\text{ELHSL}_3$  the largest subvariety of HSL with normalizers? The following lemma is explained in Kohler [6]:

### Lemma

Let  $\mathbb{V}$  be a subvariety of HSL. Then  $\mathbb{V} \subseteq \text{ELHSL}$  or  $\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \uparrow \quad \downarrow \\ \circ \end{array} \in \mathbb{V}$ .

Hence, if  $\mathbb{V} \not\subseteq \text{ELHSL}_3$ :  $\mathbf{C}_4 \in \mathbb{V}$  or  $\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \uparrow \quad \downarrow \\ \circ \end{array} \in \mathbb{V}$ .

The example on the left is given in García-Martínez et al. [3] to show that HSL does not have normalizers. It also shows that if  $\mathbf{C}_4 \in \mathbb{V}$ , then  $\mathbb{V}$  will not have normalizers, since the subalgebra on the left would then also be in  $\mathbb{V}$ .



The example on the right is a Heyting semilattice that is a subalgebra of

  $\times$   . Hence, if   $\in \mathbb{V}$ , then  $\mathbb{V}$  does not have normalizers.

# A formula for the normalizer

## Proposition

*Let  $X$  be a Heyting semilattice and  $S$  a subalgebra of  $X$ . Let  $x \in X$  be such that*

$$(S \rightarrow x) \rightarrow x \subseteq S \text{ and } (S \rightarrow x) \rightarrow S \subseteq S.$$

*Then we have that  $S$  is normal in the algebra  $\langle S, x \rangle$ .*

## Theorem

*If the normalizer exists for a subalgebra  $S$  in a given Heyting semilattice  $X$ , then the normalizer  $N_X(S)$  is given by*

$$N_X(S) = \{x \in X \mid (S \rightarrow x) \rightarrow x \subseteq S \text{ and } (S \rightarrow x) \rightarrow S \subseteq S\}.$$

In ELHSL, this formula can be simplified:

### Theorem

*If the normalizer exists for a subalgebra  $S$  in a given equationally linear Heyting semilattice  $X$ , then the normalizer  $N_X(S)$  is given by*

$$\begin{aligned} N_X(S) &= \{x \in X \mid \text{for all } s \in S, (s \rightarrow x) \rightarrow x \in S \text{ and } (s \rightarrow x) \rightarrow s \in S\} \\ &= \{x \in X \mid \text{for all } s \in S, (s \rightarrow x) \rightarrow x \in S \text{ and } (x \rightarrow s) \rightarrow s \in S\} \end{aligned}$$

In García-Martínez et al. [3], a formula for centralizers in Heyting semilattices is given.

### Groups

### ELHSL<sub>3</sub>

When subalgebra  
 $S$  normal in  $X$ ?

$$\forall x \in X, xSx^{-1} \subseteq S$$

$$\forall x \in X, (S \rightarrow x) \rightarrow x \subseteq S$$

$C_X(S)$

$$\{x \in X \mid \forall s \in S, xSx^{-1} = s\}$$

$$\{x \in X \mid \forall s \in S, \begin{matrix} (s \rightarrow x) \rightarrow x = 1 \\ \text{and } (x \rightarrow s) \rightarrow s = 1 \end{matrix}\}$$

$N_X(S)$

$$\begin{aligned} &\{x \in X \mid xSx^{-1} = S\} \\ &= \{x \in X \mid \begin{matrix} xSx^{-1} \subseteq S \\ \text{and } x^{-1}Sx \subseteq S \end{matrix}\} \end{aligned}$$

$$\{x \in X \mid \begin{matrix} \forall s \in S, (s \rightarrow x) \rightarrow x \in S \\ \text{and } (x \rightarrow s) \rightarrow s \in S \end{matrix}\}$$

# The amalgamation property

## Definition (Double extension)

A double extension in a class  $\mathcal{K}$  of algebras is a quintuple  $(A, f, B, g, C)$  with  $A, B, C \in \mathcal{K}$  and  $f : A \rightarrow B$  and  $g : A \rightarrow C$  embeddings.

## Definition (Amalgam)

An amalgam in a class  $\mathcal{K}$  of algebras of a double extension  $(A, f, B, g, C)$  is a triple  $(f', g', D)$  with  $D \in \mathcal{K}$  and  $f' : B \rightarrow D$ ,  $g' : C \rightarrow D$  embeddings, satisfying  $f' \circ f = g' \circ g$ .

- We say that a variety  $\mathbb{V}$  satisfies the **amalgamation property** if every double extension  $(A, f, B, g, C)$  in  $\mathbb{V}$  has an amalgam in  $\mathbb{V}$ .
- We say that a variety  $\mathbb{V}$  satisfies the **amalgamation property for normal monomorphisms** if every double extension  $(A, f, B, g, C)$  in  $\mathbb{V}$  with  $f$  and  $g$  normal monomorphisms, has an amalgam in  $\mathbb{V}$ .

A variety  $\mathbb{V}$  is said to have **hereditary subdirectly irreducible algebras** if for any subdirectly irreducible algebra  $A \in \mathbb{V}$ , we have that all subalgebras of  $A$  are subdirectly irreducible or trivial.

### Theorem (Theorem 3 in Gratzer and Lakser [4])

*Let  $\mathbb{V}$  be a variety that satisfies the Congruence Extension Property and that has hereditary subdirectly irreducible algebras. Then  $\mathbb{V}$  satisfies the amalgamation property if and only if every double extension  $(A, f, B, g, C)$  with  $A, B, C \in \mathbb{V}_{SI}$  has an amalgam in  $\mathbb{V}$ .*

An **ideal-determined variety** is one which is 0-regular (every congruence is determined by its 0-class) and in which all ideals are normal. (see G. Janelidze and Ursini [2] for further explanation).

### Proposition

*Let  $\mathbb{V}$  be a variety that satisfies the Congruence Extension Property and that is ideal-determined. Then  $\mathbb{V}$  satisfies the amalgamation property for normal monomorphisms if and only if every double extension  $(A, f, B, g, C)$  with  $A, B, C \in \mathbb{V}_{SI}$  and  $f, g$  normal monomorphisms, has an amalgam in  $\mathbb{V}$ .*

Some examples of varieties that are both ideal-determined and satisfy the Congruence Extension Property are rings (without unity) and Heyting semilattices.

## Corollary

*Let  $\mathbb{V}$  be a subvariety of Heyting semilattices. Then  $\mathbb{V}$  satisfies the amalgamation property for normal monomorphisms if and only if every double extension  $(A, f, B, g, C)$  with  $A, B, C \in \mathbb{V}_{SI}$  and  $f, g$  normal monomorphisms, has an amalgam in  $\mathbb{V}$ .*

## Lemma

*Let  $\mathbb{V}$  be a subvariety of Heyting semilattices with  $\mathbb{V} \subseteq \text{ELHSL}$ . Then  $\mathbb{V}$  satisfies the amalgamation property for normal monomorphisms.*

Theorem 6.5 from Borceux, Janelidze, and Kelly [1] tells us that if a semi-abelian variety  $\mathbb{V}$  satisfies the following properties:

- amalgamation property for normal monomorphisms
- has normality of unions,

then  $\mathbb{V}$  is action representable.

This allows us to deduce that  $\text{ELHSL}_3$  is action representable.

Now we can ask:

- What do actions look like in  $\text{ELHSL}_3$ ?
- How is the functor  $\text{Act}(-, X)$  representable?

- [1] F. Borceux, G. Janelidze, and G.M. Kelly. “On the representability of actions in a semi-abelian category”. In: *Theory and Applications of Categories* 14.11 (2005), pp. 244–286.
- [2] W. Tholen G. Janelidze L. Marki and A. Ursini. “Ideal determined categories”. In: *Cahiers de topologie et géométrie différentielle catégoriques* 51.2 (2010), pp. 115–125.
- [3] X. García-Martínez et al. “Categorical-algebraic aspects of Heyting semilattices”. In: (2025). DOI: 10.48550/arXiv.2508.11250.
- [4] G. Gratzer and H. Lakser. “The Structure of Pseudocomplemented Distributive Lattices. II: Congruence Extension and Amalgamation”. In: *Transaction of the American Mathematical Society* 156 (1971).
- [5] J.R.A. Gray. “Normalizers, Centralizers and Action Representability in Semi-Abelian Categories”. In: *Applied Categorical Structures* 22.5–6 (2014), pp. 981–1007. DOI: 10.1007/s10485-014-9379-2.
- [6] P. Kohler. “Brouwerian Semilattices”. In: *Transaction of the American Mathematical Society* 268 (1981).

- [7] W. Nemitz. “Implicative Semi-lattices”. In: *Transaction of the American Mathematical Society* 117 (1965), pp. 128–142.
- [8] W. Nemitz and T. Whaley. “Varieties of Implicative Semilattices”. In: *Pacific Journal of Mathematics* 37.3 (1971), pp. 759–769.