

Locales in (pre-)sheaf toposes vs. (pre-)sheaves of locales

V. Aravantinos, P. Karazeris & J. Wrigley

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(when $\mathcal{C}$ has finite limits) for every pullback

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Internal frames associated with a presheaf of frames.

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The left adjoint $\Sigma_r^{\mathfrak{z}L}$ to $\mathfrak{z}Lr$ is given by

$$\Sigma_r^{\mathfrak{z}L}(u_f) = (\bigvee_{rf=g} u_f)_g.$$

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- $\mathcal{C}$ is a groupoid.

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In terms of sections we get $\zeta_a(u) = (\downarrow Lf(u))_f$, for $I \in \text{Idl}(La)$, $\bigvee I = S_a(Lf[I])_f$ and an extra left adjoint $\lambda_a: La \rightarrow \text{Idl}(La)$ as $\lambda_a(u) = \langle \bigcup_f \Sigma_f(\Lambda_a u)_f \rangle$. Hence

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Proposition: The sections of an internally locally compact frame are locally compact. Moreover (with some more effort), for all the transitions $Lg: La \rightarrow Lb$, if $w \ll u \in La$, then $Lg(w) \ll Lg(u)$ in Lb .

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For $\mathbf{2} = \{0 \leq 1\}$ the subobject classifier $\Omega \in [\mathbf{2}^{op}, \mathbf{Set}]$ is locally compact and $\ll_{\Omega}(1) \subsetneq \ll_{\Omega_1}$.

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Proof: By the classical (left) adjoint lifting theorem

$$\begin{array}{ccc}
 \mathbf{Frm}(\widetilde{(\mathcal{C}, J)}) & \longrightarrow & \mathbf{Frm}(\widehat{\mathcal{C}}) \\
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$$\mathfrak{Z} = F \cdot \mathfrak{z} \cdot i: [\mathcal{C}^{op}, \mathbf{Frm}]_J \rightarrow [\mathcal{C}^{op}, \mathbf{Frm}] \rightarrow \mathbf{Frm}(\widehat{\mathcal{C}}) \rightarrow \mathbf{Frm}(\widetilde{(\mathcal{C}, J)})$$

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Can we transfer this to sheaves?

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- When J is generated by finite coverings and L is a frame in sheaves, then $\text{Idl}(L)$ is a sheaf.

Proof: Let $\{r_i: c_i \rightarrow c \mid i = 1, \dots, n\} \in J(c)$, I, J ideals of Lc such that $\forall i \ Lr_i[I] = Lr_i[J]$ and $u \in I$. Then

$$\forall i \quad Lr_i u \leq Lr_i v_i, \quad \text{for some } v_1, \dots, v_n \in J.$$

For $v = v_1 \vee \dots \vee v_n \in J$ we have $Lr_i(u \wedge v) = Lr_i u$, $i = 1, \dots, n$, hence $u \wedge v = u$, i.e. $u \leq v \in J$, so $I \subseteq J$.

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For a compatible family of ideals $\{K_i \mid i = 1, \dots, n\}$, i.e.

$$Lr_i^{ij} K_i = Lr_j^{ij} K_j, \quad r_i^{ij}: c_i \times_c c_j \rightarrow c_i$$

set $K = \langle \bigcup_i \Sigma_{r_i}[K_i] \rangle$.

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Corollary: For $(\mathcal{C}, J)$ finitely generated site, L a frame in sheaves, $\text{Idl}_{\widetilde{(\mathcal{C}, J)}}(L) = \mathfrak{Z}(\text{Idl} \circ L)$.

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-  G.S.H. Cruttwell, F. Marmolejo and R.J. Wood, CCD lattices in presheaf categories, *Theory Appl. Categ.*, **18**, 157–171 2007.
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