

A Swiss cheese construction for operads

Florian De Leger

National Technical University of Athens

PSSL113, Louvain-la-Neuve
27 September 2026



H.F.R.I.
Hellenic Foundation for
Research & Innovation

Overview

- 1 McClure-Smith's proof of Deligne's conjecture
- 2 Kontsevich's Swiss cheese conjecture and generalisation

Plan

- 1 McClure-Smith's proof of Deligne's conjecture
- 2 Kontsevich's Swiss cheese conjecture and generalisation

Hochschild cochain complex and Deligne's conjecture

Definition

The *Hochschild cochain complex* $\text{Hoch}(A) := C^*(A, A)$ of an associative algebra A is defined as

$$C^n(A, A) := \text{Hom}(A^{\otimes n}, A),$$

with differential $d : C^n(A, A) \rightarrow C^{n+1}(A, A)$ given by

$$\begin{aligned} d\phi(a_1 \otimes \dots \otimes a_{n+1}) &:= a_1 \cdot \phi(a_2 \otimes \dots \otimes a_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i \phi(a_1 \otimes \dots \otimes a_i \cdot a_{i+1} \otimes \dots \otimes a_{n+1}) \\ &\quad + (-1)^{n+1} \phi(a_1 \otimes \dots \otimes a_n) \cdot a_{n+1}. \end{aligned}$$

Hochschild cochain complex and Deligne's conjecture

Definition

The *Hochschild cochain complex* $\mathrm{Hoch}(A) := C^*(A, A)$ of an associative algebra A is defined as

$$C^n(A, A) := \mathrm{Hom}(A^{\otimes n}, A),$$

with differential $d : C^n(A, A) \rightarrow C^{n+1}(A, A)$ given by

$$\begin{aligned} d\phi(a_1 \otimes \dots \otimes a_{n+1}) &:= a_1 \cdot \phi(a_2 \otimes \dots \otimes a_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i \phi(a_1 \otimes \dots \otimes a_i \cdot a_{i+1} \otimes \dots \otimes a_{n+1}) \\ &\quad + (-1)^{n+1} \phi(a_1 \otimes \dots \otimes a_n) \cdot a_{n+1}. \end{aligned}$$

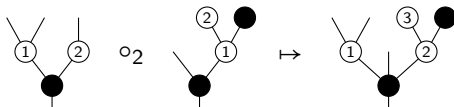
Theorem (Deligne's conjecture)

The Hochschild cochain complex of an associative algebra admits an action of (a dg-operad quasi-isomorphic to) the little 2-disks operad.

The operad $\mathcal{L}_2$ for multiplicative non-symmetric operads

Let $\mathcal{L}_2$ be the coloured operad in $\mathbf{Set}$ with set of colours $\mathbb{N}$ and $\mathcal{L}_2(n_1, \dots, n_k; n)$ is the set of (rooted) planar trees with n leaves, k white vertices with $n_1, \dots, n_k$ input edges respectively, and black vertices, such that there are no edges between two black vertices and no unary black vertices. Composition is given by inserting a tree inside a vertex then reordering, contracting edges between two black vertices and deleting unary black vertices:

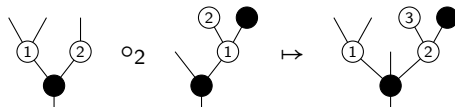
$$\mathcal{L}_2(2, 1; 3) \times \mathcal{L}_2(2, 0; 1) \rightarrow \mathcal{L}_2(2, 2, 0; 3)$$



The operad $\mathcal{L}_2$ for multiplicative non-symmetric operads

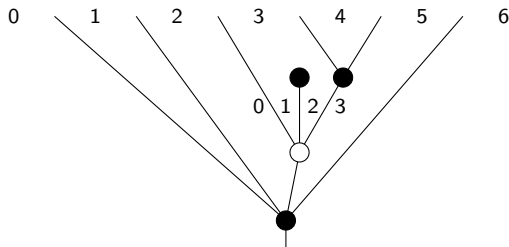
Let $\mathcal{L}_2$ be the coloured operad in $\mathbf{Set}$ with set of colours $\mathbb{N}$ and $\mathcal{L}_2(n_1, \dots, n_k; n)$ is the set of (rooted) planar trees with n leaves, k white vertices with $n_1, \dots, n_k$ input edges respectively, and black vertices, such that there are no edges between two black vertices and no unary black vertices. Composition is given by inserting a tree inside a vertex then reordering, contracting edges between two black vertices and deleting unary black vertices:

$$\mathcal{L}_2(2, 1; 3) \times \mathcal{L}_2(2, 0; 1) \rightarrow \mathcal{L}_2(2, 2, 0; 3)$$



$\mathcal{L}_2$ is the operad for *multiplicative* non-symmetric operads, that is, non-symmetric operads $\mathcal{O}$ equipped with a map $\text{Ass} \rightarrow \mathcal{O}$.

The category of unary operations of $\mathcal{L}_2$ is Δ



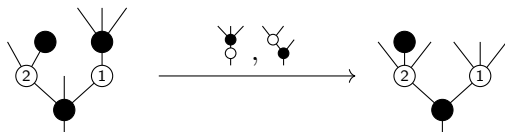
$$[3] \rightarrow [6]$$

0	2
1	3
2	3
3	5

The condensation of $\mathcal{L}_2$

For $k, n \geq 0$, let $\xi_k(n)$ be the category whose

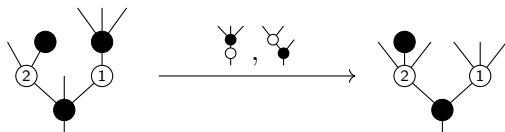
- objects are operations of $\mathcal{L}_2$ with k white vertices and n leaves,
- a morphism $p \rightarrow q$ is given by a k -tuple $f_1, \dots, f_k$ of morphisms in Δ such that $p = q \circ (f_1, \dots, f_k)$:



The condensation of $\mathcal{L}_2$

For $k, n \geq 0$, let $\xi_k(n)$ be the category whose

- objects are operations of $\mathcal{L}_2$ with k white vertices and n leaves,
- a morphism $p \rightarrow q$ is given by a k -tuple $f_1, \dots, f_k$ of morphisms in Δ such that $p = q \circ (f_1, \dots, f_k)$:



We get a cosimplicial object in the category of chain complexes:

$$\Delta \xrightarrow{\xi_k} \text{Cat} \xrightarrow{N} \text{sSet} \xrightarrow{C_*} \text{Ch}_*.$$

Let $\text{Coend}(\mathcal{L}_2)(k)$ be the *totalization* of this cosimplicial object. This defines a dg-operad $\text{Coend}(\mathcal{L}_2)$.

McClure-Smith's proof

- $\text{Hoch}(A)$ admits a $\text{Coend}(\mathcal{L}_2)$ -action :

A is an associative algebra $\iff A$ is an Ass-algebra,
 $\iff$ there is a map $\text{Ass} \rightarrow \text{End}_A$,
 $\iff \text{End}_A$ is an $\mathcal{L}_2$ -algebra,
 $\implies \text{Hoch}(A)$ is a $\text{Coend}(\mathcal{L}_2)$ -algebra.

McClure-Smith's proof

- $\text{Hoch}(A)$ admits a $\text{Coend}(\mathcal{L}_2)$ -action :

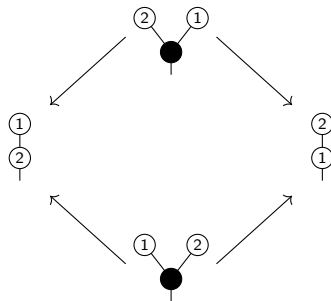
A is an associative algebra $\iff A$ is an Ass-algebra,

$\iff$ there is a map $\text{Ass} \rightarrow \text{End}_A$,

$\iff \text{End}_A$ is an $\mathcal{L}_2$ -algebra,

$\implies \text{Hoch}(A)$ is a $\text{Coend}(\mathcal{L}_2)$ -algebra.

- $\text{Coend}(\mathcal{L}_2)$ is quasi-isomorphic to the little 2-disks operad:



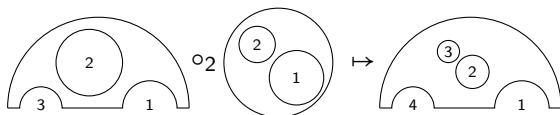
Plan

- 1 McClure-Smith's proof of Deligne's conjecture
- 2 Kontsevich's Swiss cheese conjecture and generalisation

The Swiss cheese operad and Kontsevich's conjecture

The *Swiss cheese operad* $\mathcal{SC}_m$ is the coloured operad with set of colours $\{f, h\}$ and $\mathcal{SC}_m(c_1, \dots, c_k; c)$ is a space of (ordered) configurations of (non-overlapping) little m -disks and little half m -disks inside the unit half m -disk if $c = h$, is a space of configurations of little m -disks inside the unit m -disk if $c = f$ and $c_i = f$ for all i , and is empty if $c = f$ and $c_i = h$ for some i . Composition is given by plugging an m -disk inside an m -disk or a half m -disk inside a half m -disk then reordering:

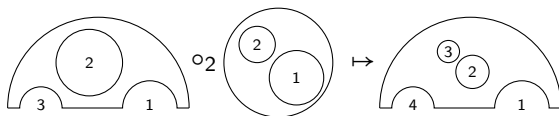
$$\mathcal{SC}_2(h, f, h; h) \times \mathcal{SC}_2(f, f; f) \rightarrow \mathcal{SC}_2(h, f, f, h; h)$$



The Swiss cheese operad and Kontsevich's conjecture

The *Swiss cheese operad* $\mathcal{SC}_m$ is the coloured operad with set of colours $\{f, h\}$ and $\mathcal{SC}_m(c_1, \dots, c_k; c)$ is a space of (ordered) configurations of (non-overlapping) little m -disks and little half m -disks inside the unit half m -disk if $c = h$, is a space of configurations of little m -disks inside the unit m -disk if $c = f$ and $c_i = f$ for all i , and is empty if $c = f$ and $c_i = h$ for some i . Composition is given by plugging an m -disk inside an m -disk or a half m -disk inside a half m -disk then reordering:

$$\mathcal{SC}_2(h, f, h; h) \times \mathcal{SC}_2(f, f; f) \rightarrow \mathcal{SC}_2(h, f, f, h; h)$$



Theorem (Kontsevich's Swiss cheese conjecture)

For an algebra A over the little m -disks operad, there is a universal object $\text{Hoch}(A)$ such that the pair $(\text{Hoch}(A), A)$ admits an $\mathcal{SC}_{m+1}$ -action.

Dolgushev-Tamarkin-Tsygan's proof

Let $\mathcal{O}$ be the operad for pairs $(\mathcal{O}, A)$, where $\mathcal{O}$ is a multiplicative non-symmetric operad and A is an $\mathcal{O}$ -algebra. Then

- the pair $(\mathrm{Hoch}(A), A)$ admits a $\mathrm{Coend}(\mathcal{O})$ -action,
- $\mathrm{Coend}(\mathcal{O})$ is quasi-isomorphic to $\mathcal{SC}_2$.

Dolgushev-Tamarkin-Tsygan's proof

Let $\emptyset$ be the operad for pairs $(\mathcal{O}, A)$, where $\mathcal{O}$ is a multiplicative non-symmetric operad and A is an $\mathcal{O}$ -algebra. Then

- the pair $(\text{Hoch}(A), A)$ admits a $\text{Coend}(\emptyset)$ -action,
- $\text{Coend}(\emptyset)$ is quasi-isomorphic to $\mathcal{SC}_2$.

Note that the operad NOp for non-symmetric operads can be obtained from Ass by applying the *Baez-Dolan plus construction*:

$$\text{Ass}^+ = \text{NOp}.$$

Dolgushev-Tamarkin-Tsygan's proof

Let $\emptyset$ be the operad for pairs $(\mathcal{O}, A)$, where $\mathcal{O}$ is a multiplicative non-symmetric operad and A is an $\mathcal{O}$ -algebra. Then

- the pair $(\text{Hoch}(A), A)$ admits a $\text{Coend}(\emptyset)$ -action,
- $\text{Coend}(\emptyset)$ is quasi-isomorphic to $\mathcal{SC}_2$.

Note that the operad NOp for non-symmetric operads can be obtained from Ass by applying the *Baez-Dolan plus construction*:

$$\text{Ass}^+ = \text{NOp}.$$

One can define a *Swiss cheese version* of the Baez-Dolan plus construction. The operad $\emptyset$ is obtained from Ass by applying this new construction:

$$\text{Ass}_{\text{sc}}^+ = \emptyset.$$

Theorem (D.)

For a coloured operad $\mathcal{P}$ and a $\mathcal{P}$ -algebra A , there is $\text{Hoch}(A)$ such that the pair $(\text{Hoch}(A), A)$ admits an action of $\text{SC}(\mathcal{P}) := \text{Coend}(\mathcal{P}_{sc}^+)$.

New results

Theorem (D.)

For a coloured operad $\mathcal{P}$ and a $\mathcal{P}$ -algebra A , there is $\text{Hoch}(A)$ such that the pair $(\text{Hoch}(A), A)$ admits an action of $\text{SC}(\mathcal{P}) := \text{Coend}(\mathcal{P}_{sc}^+)$.

Theorem (D.)

There is a weak equivalence of categorical operads $\text{SC}(\mathcal{K}_m) \rightarrow \mathcal{RK}_{m+1}$, where $\mathcal{K}_m$ is Berger's *complete graph operad* and $\mathcal{RK}_m$ is Quesney's *extended complete graph operad*.

New results

Theorem (D.)

For a coloured operad $\mathcal{P}$ and a $\mathcal{P}$ -algebra A , there is $\text{Hoch}(A)$ such that the pair $(\text{Hoch}(A), A)$ admits an action of $\text{SC}(\mathcal{P}) := \text{Coend}(\mathcal{P}_{sc}^+)$.

Theorem (D.)

There is a weak equivalence of categorical operads $\text{SC}(\mathcal{K}_m) \rightarrow \mathcal{RK}_{m+1}$, where $\mathcal{K}_m$ is Berger's *complete graph operad* and $\mathcal{RK}_m$ is Quesney's *extended complete graph operad*.

Corollary (Ginot-Tradler-Zeinalian)

The Hochschild-Pirashvili cochain complex of order m of a commutative algebra admits an action of the little $m + 1$ -disks operad.

References

- J. Baez and J. Dolan, *Higher-dimensional algebra III. n -categories and the algebra of opetopes*, Adv. Math., 135(2):145–206, 1998.
- M. Batanin and C. Berger, *The lattice path operad and Hochschild cochains*, Contemp. Math., 504:23–52, 2009.
- C. Berger, *Combinatorial models for real configuration spaces and E_n -operads*, Contemp. Math., 202:37–52, 1997.
- F. De Leger, *A combinatorial approach to Kontsevich's Swiss cheese conjecture*, arXiv preprint arXiv:2512.20167.
- V. Dolgushev, D. Tamarkin, and B. Tsygan, *Proof of Swiss cheese version of Deligne's conjecture*, Int. Math. Res. Not., 2011(20):4666–4746, 2011.
- G. Ginot, T. Tradler, and M. Zeinalian, *Higher Hochschild cohomology, Brane topology and centralizers of E_n -algebra maps*, arXiv preprint arXiv:1205.7056.
- M. Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys., 48(1), 35–72.
- J. McClure and J. Smith, *Cosimplicial objects and little n -cubes, I*, Am. J. Math., 126(5):1109–1153, 2004.
- T. Pirashvili, *Hodge decomposition for higher order Hochschild homology*, Ann. Sci. Éc. Norm. Supér., 33:151–179, 2000.
- A. Quesney, *The relative lattice path operad*, Algebr. Geom. Topol., 18(3):1753–1798, 2018.
- A. Voronov, *The Swiss-cheese operad*, Contemp. Math., 239(11):365–374: 1999.