

Generalized action monads and descent

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Action monads

Let $\mathcal{C}$ be a category with finite limits.

Given a group object $G \in \mathcal{C}$, its **action monad** is the monad

$$\begin{array}{ccc} \mathcal{T}_G : & \mathcal{C} & \longrightarrow \mathcal{C} \\ & \begin{array}{ccc} \mathcal{C} & & G \times \mathcal{C} \\ \downarrow f & \mapsto & \downarrow \text{id}_G \times f \\ \mathcal{C}' & & G \times \mathcal{C}' \end{array} \end{array}$$

with multiplication given by $G \times (G \times C) \cong (G \times G) \times C \xrightarrow{m_G \times \text{id}_C} G \times C$
and unit given by $C \cong 1 \times C \xrightarrow{e_G \times \text{id}_C} G \times C$.

$$\text{Alg}(\mathcal{T}_G) \cong \mathbf{Act}_{\mathcal{C}}(G)$$

Actions of internal categories

Definition (G. Janelidze–W. Tholen).

A **(left) action** of the internal category $\mathbb{D} = (D_1 \overset{s}{\underset{t}{\rightrightarrows}} i D_0)$ is a triple (C, γ, ξ) consisting of

- an object C of $\mathcal{C}$
- a morphism $C \xrightarrow{\gamma} D_0$
- a morphism $D_1 \times_{D_0}^{\gamma, s} C \xrightarrow{\xi} C$

such that

$$\begin{array}{ccccc}
 D_1 \times_{D_0} C & \xrightarrow{\xi} & C & \xrightarrow{\langle i \circ \gamma, \text{id}_X \rangle} & D_1 \times_{D_0} C & \xrightarrow{\text{comp} \times \text{id}_X} & D_1 \times_{D_0} D_1 \times_{D_0} C & \xrightarrow{\text{id}_{D_1} \times \text{comp}} & D_1 \times_{D_0} C & \xrightarrow{\xi} & C \\
 \pi_1 \downarrow & & \downarrow \gamma & \searrow & \downarrow \xi & & \downarrow \xi & & \downarrow \xi & & \\
 D_1 & \xrightarrow{t} & D_0 & & C & & D_1 \times_{D_0} C & \xrightarrow{\xi} & C & &
 \end{array}$$

Generalized action monads

$$\mathcal{T}_{\mathbb{D}} : \quad C/D_0 \longrightarrow C/D_0$$

$$\begin{array}{ccc}
 C & & D_1 \times_{D_0} C \\
 \downarrow \gamma & \mapsto & \downarrow \pi_1 \quad \lrcorner \quad \downarrow \gamma \\
 D_0 & & D_1 \xrightarrow{s} D_0 \\
 & & \downarrow t \\
 & & D_0
 \end{array}$$

$$\begin{array}{ccc}
 C \xrightarrow{f} C' & & D_1 \times_{D_0} C \xrightarrow{\text{id} \times f} D_1 \times_{D_0} C' \\
 \searrow \gamma \quad \swarrow \gamma' & \mapsto & \begin{array}{ccc} \pi_1 \downarrow & & \downarrow \pi_1 \\ D_1 & \equiv & D_1 \\ t \downarrow & & \downarrow t \\ D_0 & \equiv & D_0 \end{array} \\
 & & D_0
 \end{array}$$

Generalized action monads

Theorem.

The functor $\mathcal{T}_{\mathbb{D}}$ extends to a monad.

Moreover, we have an isomorphism of categories $\mathcal{Alg}(\mathcal{T}_{\mathbb{D}}) \cong \mathbf{Act}_C(\mathbb{D})$

$$\begin{array}{ccccc}
 (D_1 \times_{D_0} D_1) \times_{D_0} C & & \xrightarrow{\pi_2} & & C \\
 \downarrow \pi_1 & \searrow [\mu_{\gamma}^{\mathbb{D}}] & & \searrow \pi_2 & \downarrow \gamma \\
 D_1 \times_{D_0} D_1 & & D_1 \times_{D_0} C & \xrightarrow{\pi_2} & C \\
 & \searrow \text{comp} & \downarrow \pi_1 & & \downarrow \gamma \\
 & & D_1 & \xrightarrow{s} & D_0
 \end{array}$$

$$\begin{array}{ccccc}
 C & & \xrightarrow{\pi_2} & & C \\
 \downarrow \gamma & \searrow \eta_{\gamma}^{\mathbb{D}} & & \searrow \pi_2 & \downarrow \gamma \\
 D_0 & \xrightarrow{i} & D_1 & \xrightarrow{s} & D_0 \\
 & \searrow & \downarrow t & & \\
 & & D_0 & &
 \end{array}$$

Comorphisms of monads

Definition.

A *comorphism of monads* $(\mathcal{E}, T) \longrightarrow (\mathcal{C}, S)$ consists of a pair (F, χ) where $F : \mathcal{E} \rightarrow \mathcal{C}$ is a functor and χ is a natural transformation $\chi : FT \Rightarrow SF$ compatible with the units and multiplications of the two monads.

In general, (F, χ) does not induce a functor between $\mathcal{Alg}(T)$ and $\mathcal{Alg}(S)$.

But if $F \dashv G$, the natural transformation $\chi : FT \Rightarrow SF$ has a mate $\chi^\sharp : TG \Longrightarrow GS$, such that $(G, \chi^\sharp)$ is a morphism of monads.

So (F, χ) induces a functor

$$\mathcal{Alg}((F, \chi)) : \mathcal{Alg}(S) \longrightarrow \mathcal{Alg}(T)$$

sending an S -algebra (Y, b) to the T -algebra $(GY, Gb \circ \chi_Y^\sharp)$.

The functoriality of $\mathbf{Act}_C(-)$

Theorem.

The assignment $\mathbb{D} \mapsto \mathcal{T}_{\mathbb{D}}$ extends to a functor $\mathcal{T} : \mathbf{Cat}(C) \rightarrow \mathcal{Mnd}_{\text{co}}^{\text{adj}}$

Theorem.

The assignment $\mathbb{D} \mapsto \mathbf{Act}_C(\mathbb{D})$ extends to a functor

$$\mathbf{Act}_C(-) : \mathbf{Cat}(C)^{\text{op}} \rightarrow \mathbf{Cat}.$$

More precisely, $\mathbf{Act}_C(-)$ is given by the composite of functors

$$\mathbf{Cat}(C)^{\text{op}} \xrightarrow{\mathcal{T}^{\text{op}}} (\mathcal{Mnd}_{\text{co}}^{\text{adj}})^{\text{op}} \xrightarrow{\mathcal{Alg}(-)} \mathbf{Cat}$$

Properties and structures of $\mathcal{T}_{\mathbb{D}}$

The monad $\mathcal{T}_{\mathbb{D}}$ is naturally equipped with a natural transformation

$$\begin{array}{ccc} C/D_0 & \xrightarrow{\mathcal{T}_{\mathbb{D}}} & C/D_0 \\ \text{dom} \downarrow & \xleftarrow{\alpha} & \swarrow \text{dom} \\ & C & \end{array}$$

whose component on $\gamma: C \rightarrow D_0$ is the projection $D_1 \times_{D_0} C \xrightarrow{\pi_2} C$.

- α is *cartesian*
- α is compatible with $\eta^{\mathbb{D}}$ and $\mu^{\mathbb{D}}$. Equivalently, we have a comorphism of monads $(\text{dom}, \alpha): (C/D_0, \mathcal{T}_{\mathbb{D}}) \rightarrow (C, \text{Id}_C)$.

Notice that (dom, α) induces a functor

$$C \cong \mathcal{Alg}(\text{Id}_C) \rightarrow \mathcal{Alg}(\mathcal{T}_{\mathbb{D}}).$$

The category of generalized action monads

The category $\mathcal{ActMnd}(\mathcal{C})$ is given by the following data:

- *objects*: triples $(D_0, \mathbf{T}, \alpha)$ where $D_0 \in \mathcal{C}$ and $\mathbf{T}$ is a monad on $\mathcal{C}/D_0$ such that $\text{dom}: \mathcal{C}/D_0 \rightarrow \mathcal{C}$ extends to a comorphism of monads $(\text{dom}, \alpha): \mathbf{T} \rightarrow \text{Id}_{\mathcal{C}}$ and α is a cartesian natural transformation;
- *morphisms* $(D_0, \mathbf{T}, \alpha) \rightarrow (E_0, \mathbf{T}', \beta)$: pairs (h, λ) where $h: D_0 \rightarrow E_0$ is a morphism in $\mathcal{C}$ such that $h \circ -: \mathcal{C}/D_0 \rightarrow \mathcal{C}/E_0$ extends to a comorphism of monads $(h \circ -, \lambda): \mathbf{T} \rightarrow \mathbf{T}'$ and moreover

$$\begin{array}{ccc} \mathbf{T} & & \\ \downarrow (h \circ -, \lambda) & \searrow (\text{dom}, \alpha) & \\ \mathbf{T}' & \nearrow (\text{dom}, \beta) & \text{Id}_{\mathcal{C}} \end{array}$$

Characterizing generalized action monads

Theorem.

The functor $\mathcal{T}: \mathbf{Cat}(C) \rightarrow \mathcal{Mnd}_{co}^{adj}$ factors through $\mathcal{ActMnd}(C) \hookrightarrow \mathcal{Mnd}_{co}^{adj}$.

Moreover, we have an equivalence of categories

$$\mathcal{T}: \mathbf{Cat}(C) \simeq \mathcal{ActMnd}(C): \Psi.$$

The functor Ψ sends $(D_0, \mathbf{T}, \alpha)$ to the internal category

$$\mathrm{dom}(T^2(\mathrm{id}_{D_0})) \xrightarrow{\mu_{\mathrm{id}_{D_0}}} \mathrm{dom}(T(\mathrm{id}_{D_0})) \begin{array}{c} \xrightarrow{\alpha_{\mathrm{id}_{D_0}}} \\ \xleftarrow{\eta_{\mathrm{id}_{D_0}}} \\ \xrightarrow{T(\mathrm{id}_{D_0})} \end{array} D_0$$

The general fibrational setting

Let $p: \mathcal{E} \rightarrow \mathcal{C}$ be a (normal) bifibration satisfying the Beck–Chevalley condition at every pullback square.

We build the functor $\mathcal{T}_{\mathbb{D}}^p: p^{-1}(D_0) \rightarrow p^{-1}(D_0)$ as the composite

$$p^{-1}(D_0) \xrightarrow{s^*} p^{-1}(D_1) \xrightarrow{t_!} p^{-1}(D_0)$$

$$\begin{array}{ccc}
 s^*(\gamma) & \xrightarrow{\overline{s}^\gamma} & \gamma \\
 \downarrow p & \searrow s^* \gamma \overline{t} & \downarrow p \\
 & t_!(s^*(\gamma)) & \\
 D_1 & \xrightarrow{s} & D_0 \\
 & \searrow t & \downarrow p \\
 & & D_0
 \end{array}$$

The monad $\mathcal{T}_{\mathbb{D}}^p$

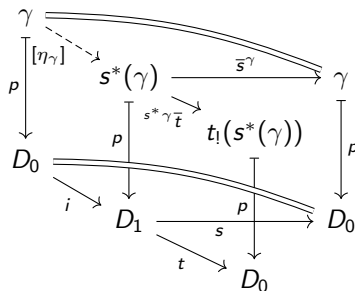
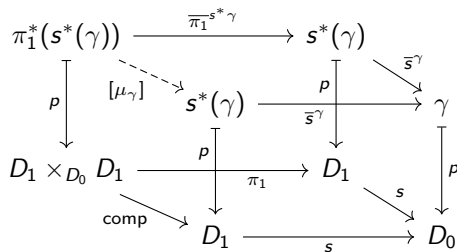
Theorem.

The functor

$$\mathcal{T}_{\mathbb{D}}^p : p^{-1}(D_0) \longrightarrow p^{-1}(D_0)$$

$$\gamma \mapsto t_!(s^*(\gamma))$$

extends to a monad $\mathcal{T}_{\mathbb{D}}^p = (\mathcal{T}_{\mathbb{D}}^p, \eta^{\mathbb{D}}, \mu^{\mathbb{D}})$.



Theorem.

If the opfibration structure of p is split, the assignment $\mathbb{D} \mapsto \mathcal{T}_{\mathbb{D}}^p$ extends to a functor $\mathcal{T}^p: \mathbf{Cat}(C) \rightarrow \mathcal{Mnd}_{\text{co}}^{\text{adj}}$.

The composite

$$\mathbf{Cat}(C)^{\text{op}} \xrightarrow{(\mathcal{T}^p)^{\text{op}}} (\mathcal{Mnd}_{\text{co}}^{\text{adj}})^{\text{op}} \xrightarrow{\mathcal{Alg}(-)} \mathbf{Cat}$$

provides a functor $\mathbf{Act}_C^p(-): \mathbf{Cat}(C)^{\text{op}} \rightarrow \mathbf{Cat}$ that takes categories of actions with respect to p .

Fibrational actions of internal categories

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provides a functor $\mathbf{Act}_C^p(-): \mathbf{Cat}(C)^{\text{op}} \rightarrow \mathbf{Cat}$ that takes categories of actions with respect to p .

It seems hard to reconstruct the internal category from the monad in the fibrational setting.

conjecture: a fibrational generalization of internal categories could work.

Descent along a morphism

Let $\mathcal{C}$ be a category with pullbacks, and let $f: E \rightarrow B$ be a morphism in $\mathcal{C}$. Consider the pullback functor $f^*: \mathcal{C}/B \rightarrow \mathcal{C}/E$.

The adjunction $f_! \dashv f^*$ gives rise to a monad $T_f = f^* f_!: \mathcal{C}/E \rightarrow \mathcal{C}/E$ and T_f -algebras are seen as descent data $\mathcal{D}esc(f) = \mathcal{A}lg(T_f)$

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We have a *comparison functor* $K_f: \mathcal{C}/B \rightarrow \mathcal{D}esc(f)$ which has a left adjoint called *realization functor*.

Definition.

The morphism $f: E \rightarrow B$ is called:

- a *descent morphism* if K_f is fully faithful;
- an *effective descent morphism* if K_f is an equivalence.

A generalization of this is *monadic descent* [Beck, Benabou-Roubaud].

Descent for a left adjoint right T -module

Definition.

Let $T = (T, \eta, \mu)$ be a monad on $\mathcal{E}$.

A *right T -module with values in $\mathcal{C}$* is a pair (L, α) , where $L : \mathcal{E} \rightarrow \mathcal{C}$ is a functor and $\alpha : LT \Rightarrow L$ is a natural transformation satisfying

$$\alpha L\eta = \text{Id}_L, \quad \alpha L\mu = \alpha\alpha_T.$$

We call (L, α) a *left adjoint right T -module* if L is a left adjoint.

A right T -module structure $\alpha : LT \Rightarrow L$ is equivalently a comorphism of monads $(L, \alpha) : (\mathcal{E}, T) \longrightarrow (\mathcal{C}, \text{Id}_{\mathcal{C}})$.

Remark.

Every generalized action monad (D_0, T, α) on $\mathcal{C}$ determines a left adjoint right T -module structure on the pair (dom, α) , where $\text{dom} : \mathcal{C}/D_0 \rightarrow \mathcal{C}$.

The comparison functor

Proposition.

Let $T = (T, \eta, \mu)$ be a monad on $\mathcal{E}$, and let $L : \mathcal{E} \rightleftarrows \mathcal{C} : R$ be an adjunction. Take RL , with its induced monad structure.

Then the following data are equivalent:

- a right T -module structure on L ;
- a left T -module structure on R ;
- a comorphism of monads $(\text{id}_{\mathcal{E}}, \theta : T \longrightarrow RL)$;
- a lift of R through the Eilenberg–Moore forgetful functor $U : \mathcal{Alg}(T) \rightarrow \mathcal{E}$, that is a functor $K_{\alpha} : \mathcal{C} \rightarrow \mathcal{Alg}(T)$ such that $U \circ \tilde{R} = R$.

The functor $K_{\alpha} : \mathcal{C} \rightarrow \mathcal{Alg}(T)$ is the one induced on algebras by the comorphism of monads of the T -module structure.

The realization functor

We construct the functor

$$\Phi_{\alpha} : \mathcal{Alg}(T) \longrightarrow \mathcal{C}.$$

sending an algebra (X, a) to the coequalizer

$$LTX \begin{array}{c} \xrightarrow{La} \\ \xrightarrow{\alpha_X} \end{array} LX \xrightarrow{q_{(X,a)}} L \otimes_T (X, a)$$

Φ_{α} is the *realization functor* associated with the right T -module (L, α) .

Theorem.

There is an adjunction $\Phi_{\alpha} \dashv K_{\alpha}$.

Recovering descent along a morphism

We can recover the classical descent along a morphism $f: E \rightarrow B$ by considering the Čech internal groupoid

$$\mathrm{Eq}(f) = (E \times_B E \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} E)$$

The obtained generalized action monad is $\mathcal{T}_{\mathrm{Eq}(f)} = f^* f_!$.

Therefore, the algebras for $\mathcal{T}_{\mathrm{Eq}(f)}$ are precisely the classical descent data.

The comparison functor and the realization functor associated to the left adjoint right $\mathcal{T}_{\mathrm{Eq}(f)}$ -module corresponding to the Čech groupoid coincide with the classical ones.

Recovering Galois descent [Marques]

Let G be an internal group over B , and suppose that G acts on $p: E \rightarrow B$ by a morphism $h: G \times_B E \rightarrow E$ over B .

This action gives rise to an action groupoid

$$G \ltimes E = \left(G \times_B E \begin{array}{c} \xrightarrow{\pi_2} \\ \xrightarrow{h} \end{array} E \right)$$

The corresponding generalized action monad is

$$T_G = \mathcal{T}_{G \ltimes E} = h_! \pi_2^*: C/E \rightarrow C/E$$

$$\mathcal{D}esc_G(E) = \mathbf{Act}_C(G \ltimes E) \simeq \mathcal{Alg}(T_G).$$

The comparison functor and the realization functor coincide with the Galois descent ones.

Comparison with monadic descent

The adjunction $L \dashv R$ gives a canonical right RL -module structure on L $\varepsilon_L: LRL \Rightarrow L$, with associated comparison functor

$$K_{\varepsilon L}: \mathcal{C} \rightarrow \mathcal{Alg}(RL), \quad K_{\varepsilon L}(C) = (RC, R\varepsilon_C),$$

which is the usual comparison functor of the adjunction $L \dashv R$.

The comorphism of monads $(\text{id}_{\mathcal{E}}, \theta: T \longrightarrow RL)$ corresponding to the right T -module structure $\alpha: LT \xrightarrow{L\theta} LRL \xrightarrow{\varepsilon L} L$ induces a functor

$$\mathcal{Alg}(\text{id}_{\mathcal{E}}, \theta): \mathcal{Alg}(RL) \rightarrow \mathcal{Alg}(T)$$

Proposition.

$$K_{\alpha} = \mathcal{Alg}(\text{id}_{\mathcal{E}}, \theta) K_{\varepsilon L}$$

The rigidity theorem for effective descent

Theorem.

Let (L, α) be a right T -module structure on L with associated comorphism of monads

$$\theta : T \rightarrow RL$$

over $\mathcal{E}$. Assume that the coequalizers defining Φ_α exist.

Then the following are equivalent:

- the comparison functor $K_\alpha : \mathcal{C} \rightarrow \mathcal{Alg}(T)$ is an equivalence;*
- the usual comparison functor $K_{\varepsilon L} : \mathcal{C} \rightarrow \mathcal{Alg}(RL)$ of the adjunction $L \dashv R$ is an equivalence, and $\theta : T \rightarrow RL$ is an isomorphism of monads.*

A sweet spot for descent

