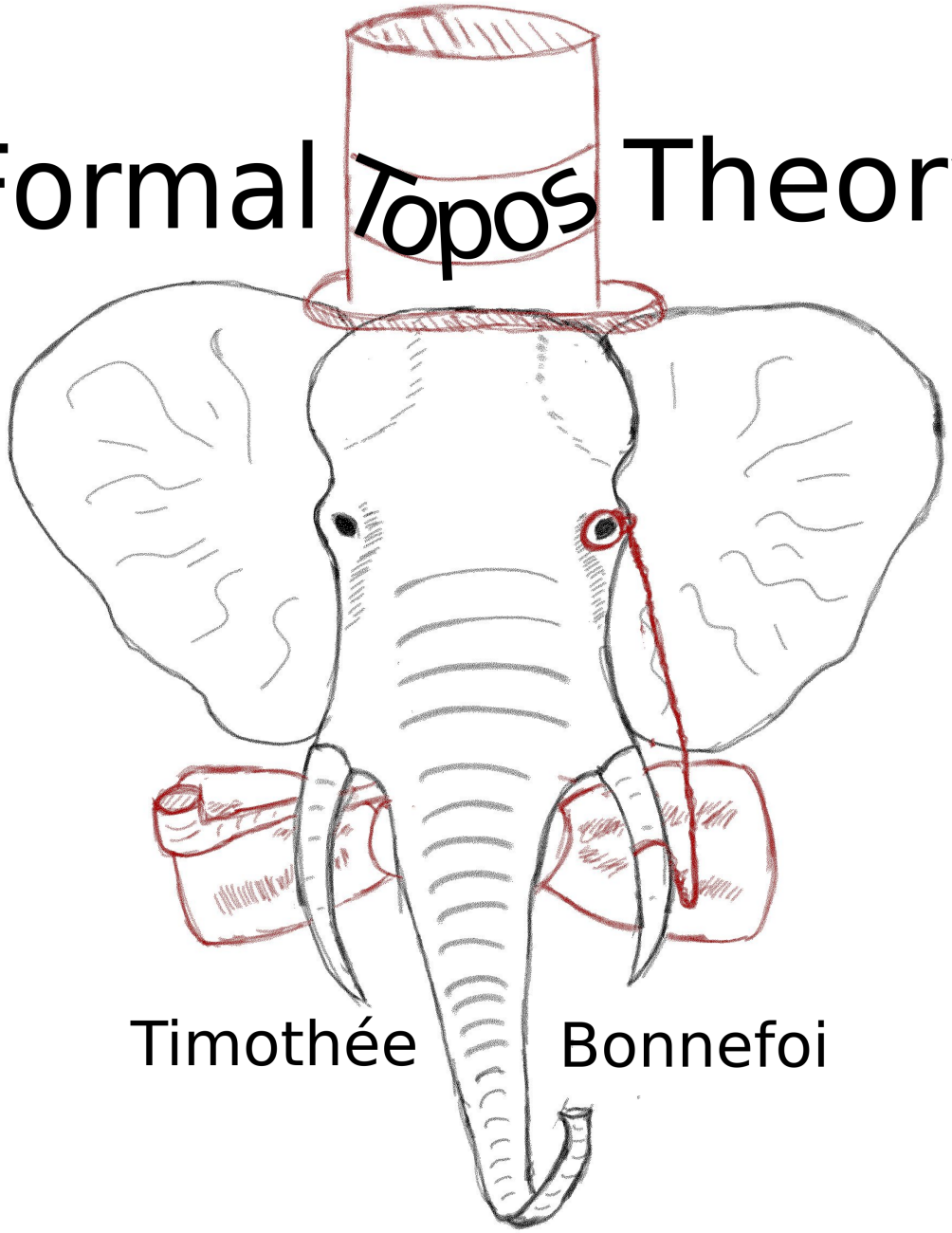
Formal Topos Theory



Timothée Bonnefoi

University of Antwerp

Formal category theory

Formal category theory = category theory in a 2-category $\mathcal{B}$
↳Category theory = Formal category theory in $\mathcal{C}at$

category $\mathrm{C} \in \mathcal{C}at$ $\leftrightsquigarrow$ object $\mathrm{E} \in \mathcal{B}$

object $\mathbf{A} \in \mathrm{C}$ $\leftrightsquigarrow$ 1-morphism $\mathrm{I} \to \mathrm{E}$

morphism $\mathbf{f} : \mathbf{A} \to \mathbf{B} \in \mathrm{C}$ $\leftrightsquigarrow$ triangle

$$\begin{tikzcd} \mathrm{I} \arrow[dd] \arrow[dr] \arrow[ddr, phantom, "\Downarrow"] & \\ & \mathrm{E} \\ \mathrm{J} \arrow[ur] & \end{tikzcd}$$

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Easy

- Adjunctions
- Monads

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Hard(-ish)

- (Co)limits
- Presheaves

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- Toposes

The elephant (Johnstone 2002)

1. A topos is a category of sheaves on a site.
2. A topos is a left-exact reflective subcategory of the category of presheaves of a small category.
3. A topos is the category of algebras of a left-exact idempotent monad on the category of presheaves of a small category.
4. A topos is the category of sheaves for a Lawvere-Tierney topology on the category of presheaves of a small category.
5. A topos is the localisation of the category of presheaves of a small category at a class of morphisms satisfying the right Ore conditions
6. A topos is locally presentable category satisfying the Giraud conditions
7. A topos is lex total category with a small set of generators.

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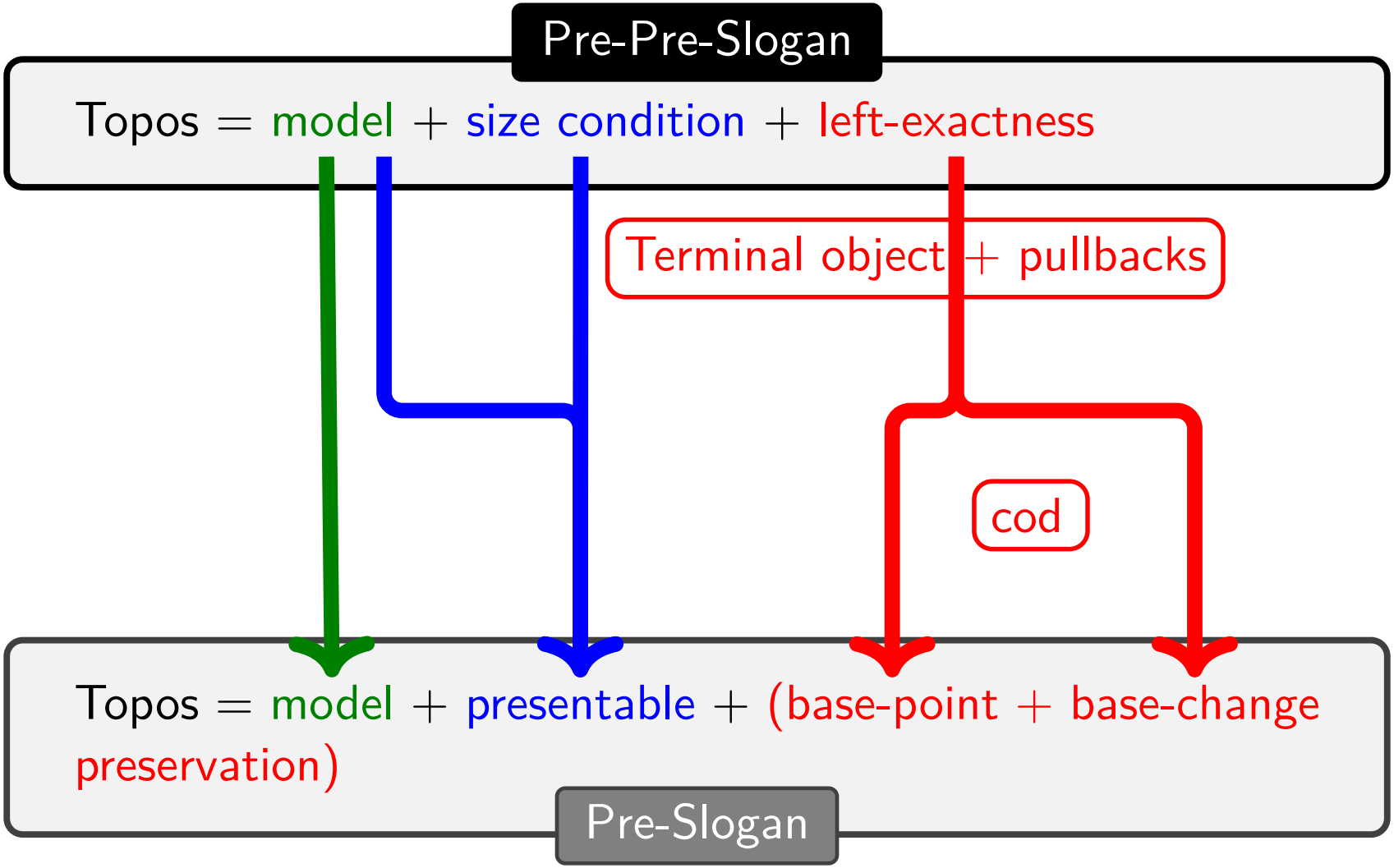
Slogans

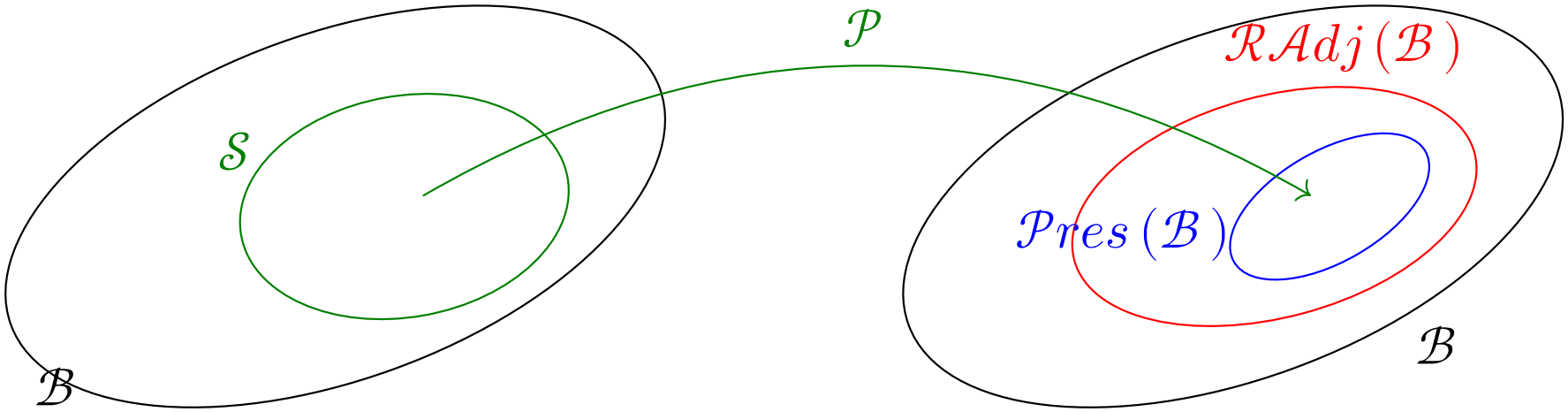
Pre-Pre-Slogan

Topos = model + size condition + left-exactness

Slogans


Pre-Pre-Slogan
Topos = model + size condition + left-exactness
Terminal object + pullbacks
cod
Topos = model + presentable + (base-point + base-change preservation)
Pre-Slogan

Models and presentability: the coarse idea



Models and presentability: examples

2-category	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{C}at$	$\mathcal{CAT}$	$\mathcal{P}rod$	$\mathcal{H}o(\mathcal{QC}at)$[1]
Models	KZ-monad $\mathcal{M}$ (Kock 1995; Zöberlein 1976)	Yoneda structure $\mathcal{Y}$ (Street and Walters 1978)	small presheaves $\mathcal{P}$	presheaves (if locally small) $\mathcal{P}r$	downward closed sets $\mathcal{D}$	presheaves $\mathcal{P}s$
strong size conditions	KZ-context (Di Liberti and Loregian 2023)		locally κ-presentable, κ fixed			κ-presentable, κ fixed
mild size conditions	cocomplete object that are subobjects of $\mathcal{M}\,\mathrm{S}$, S $\mathcal{M}$-small	total objects that are subobjects of $\mathcal{Y}\,\mathrm{S}$, S $\mathcal{Y}$-small	reflective subcategories of $\mathcal{P}\,\mathrm{S}$, S small	total categories that are subobjects of $\mathcal{P}r\,\mathrm{S}$	reflective subpreorder of $\mathcal{D}\,\mathrm{P}$	reflective subquasicategory of $\mathcal{P}s\,\mathrm{S}$
no size conditions	cocomplete object + right adjoints	total objects		total categories	reflective subpreorder of $\mathcal{D}\,\mathrm{P}$	

[1] Joint work (in progress) with Arne Mertens.

Models and presentability: exan

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$\mathcal{B}$

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$\mathcal{C}at$

small presheaves $\mathcal{P}$

locally κ-presentable, κ fixed

reflective subcategories of $\mathcal{P}\,\mathrm{S}$, S small

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and presentability: examples

		$\mathcal{C}at$	$\mathcal{CAT}$	$\mathcal{P}rod$	$\mathcal{H}o(\mathcal{QC}at)$[1]
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n c t	objects		total categories	reflective sub-preorder of $\mathcal{D}\,\mathrm{P}$	

$\mathcal{CAT}$

presheaves (if locally small) $\mathcal{P}r$

total categories that are subob- jects of $\mathcal{P}r\,\mathrm{S}$

total categories

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s s c t			locally κ-presentable, κ fixed			κ-presentable, κ fixed
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$\mathcal{P}rod$
downward closed sets $\mathcal{D}$
reflective sub-preorder of $\mathcal{D}\,\mathrm{P}$

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and presentability: examples

$\mathcal{H}o(\mathcal{QC}at)$[1]
presheaves $\mathcal{P}s$
κ-presentable, κ fixed
reflective sub-quasicategory of $\mathcal{P}s\,\mathrm{S}$

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Left-exactness from base-change

Theorem (Left-exactness through the codomain fibration)

Let $\mathrm{F} : \mathrm{C} \to \mathrm{D}$ *be a functor between categories with finite limits.*
F *is left-exact if and only if we can complete, for*
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Indexed objects $\mathrm{C}^{op} \to \mathcal{B}$

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\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \mathrm{D} \arrow[dl] \arrow[dr] & \\
 & \mathrm{C} \arrow[ur, "\mathrm{F}"] \arrow[dl] \arrow[dr] & \mathrm{D}/\mathrm{F}\mathbf{A} & \mathrm{D}/\mathrm{F}\mathbf{B} \arrow[l, "(\mathrm{F}\mathbf{f})^*"'] \\
\mathrm{C}/\mathbf{A} \arrow[ur, dotted, "\mathrm{F}(-)"] & & \mathrm{C}/\mathbf{B} \arrow[ul, "\mathbf{f}^*"] \arrow[ur, dotted, "\mathrm{F}(-)"'] &
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Left-exactness from base-change

Theorem (Left-exactness through the codomain fibration)

Let $\mathrm{F} : \mathrm{C} \to \mathrm{D}$ *be a functor between categories with finite limits.*
F *is left-exact (pullbacks) if and only if we can complete, for* $\mathbf{f} : \mathbf{A} \to \mathbf{B} \in \mathrm{C}$*:*

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 & \mathrm{C}/\mathbf{A} \arrow[ur, dotted, "\mathrm{F}(-)"] & \simeq & \\
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Self-indexing

Q: How to generalise the codomain fibration ?

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A: With a universal property of slices...but which?

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Not with commas

- enriched commas are augmented $\to$comma at terminal object is not the category itself.
- higher slice is not (but almost) a comma
- for monoidal categories and colax functors, the comma is $\sqcup_{n\in\mathbb{N}}\mathrm{C}/(\mathbf{A}^{\otimes n})$

Polynomial objects

Theorem (Artin, Grothendieck, and Verdier 1971; Lambek and Scott 1988; Hermida and Jacobs 1995)

Let C *be a finitely complete category,* $\mathbf{A} \in \mathrm{C}$ *then* $\mathrm{C}/\mathbf{A}$ *satisfies the following universal property:*

$$\begin{tikzcd}
\mathrm{C} \arrow[r, "\mathrm{H}\ \textit{lex}"'] \arrow[dr, "\mathrm{M}\ \textit{lex}"'] & \mathrm{C}/\mathbf{A} \arrow[d, dotted, "\exists!\textit{lex}"'] \\
& \mathrm{D}\ \textit{lex}
\end{tikzcd}
\qquad
\begin{tikzcd}
\chi : \mathrm{H}\mathbf{1} \arrow[r] \arrow[d, mapsto] & \mathrm{H}\mathbf{A} \\
\psi : \mathrm{M}\mathbf{1} \arrow[r] & \mathrm{M}\mathbf{A}
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Motivation: The same theory but with an extra constant/variable of sort/type $\mathbf{A}$ *.*

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& \mathrm{D} \text{ presentable} \arrow[u, bend left]
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Theorem (Left-exactness through the codomain fibration)

Let $\mathrm{F} : \mathrm{C} \to \mathrm{D}$ *be a functor between categories with finite limits.* F *is left-exact (pullbacks+terminal object) if and only if we can complete, for* $\mathbf{f} : \mathbf{A} \to \mathbf{B} \in \mathrm{C}$*:*

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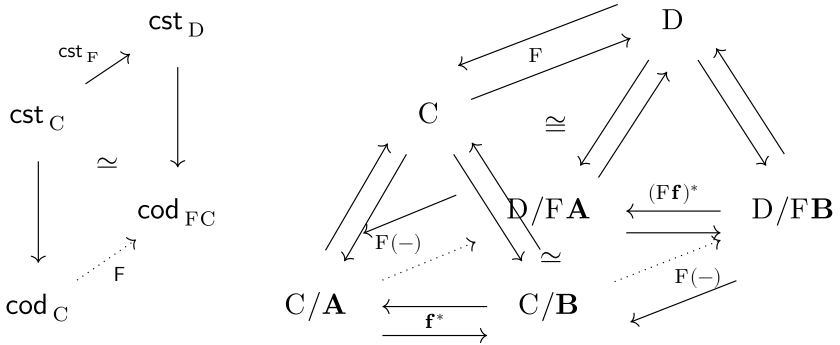
 & & \mathrm{D} \arrow[dl, shift right] \arrow[dr, shift left] \arrow[dl, shift left] \arrow[dr, shift right] \\
 & \mathrm{C} \arrow[ur, "\mathrm{F}"] \arrow[dl, shift right] \arrow[dr, shift left] \arrow[r, phantom, "\cong"] & \mathrm{D}/\mathrm{F}\mathbf{A} \arrow[r, shift left, "(\mathrm{F}\mathbf{f})^*"'] \arrow[r, phantom, "\simeq"] & \mathrm{D}/\mathrm{F}\mathbf{B} \arrow[l, shift left] \arrow[dl, dotted, "\mathrm{F}(-)"] \\
\mathrm{C}/\mathbf{A} \arrow[ur, shift right] \arrow[ur, dotted, "\mathrm{F}(-)"] & & \mathrm{C}/\mathbf{B} \arrow[ul, shift left] \arrow[ul, shift right] \arrow[ll, shift right, "\mathbf{f}^*"'] &
\end{tikzcd}$$

Indexed objects $\mathrm{C}^{op} \to \mathcal{B}$

Left-exactness from base-change

Theorem (Left-exactness through the codomain fibration)

Let $F : C \to D$ *be a functor between categories with finite limits. F is left-exact (pullbacks+terminal object) if and only if we can complete, for* $\mathbf{f} : \mathbf{A} \to \mathbf{B} \in C$*:*



Indexed objects $C \to \mathcal{P}res(\mathcal{B})$

Left-exactness from base-change

Slogan

Topos theory = Models + presentability + (base-point + base-change + slicing)

$$\begin{tikzcd}
 & \mathsf{cst}_{\mathrm{D}} \arrow[d] \\
\mathsf{cst}_{\mathrm{C}} \arrow[ur, "\mathsf{cst}_{\mathrm{F}}"] \arrow[d] \arrow[dr, phantom, "\simeq"] & \\
\mathsf{cod}_{\mathrm{C}} \arrow[r, dotted, "\mathsf{F}"] & \mathsf{cod}_{\mathrm{FC}}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathrm{D} \arrow[dl, shift right, "\mathrm{F}"'] \arrow[dl, shift left] \arrow[dd, shift right] \arrow[dd, shift left] \arrow[dr, shift right] \arrow[dr, shift left] & \\
 & \mathrm{C} \arrow[ur, shift right] \arrow[ur, shift left] \arrow[dl, shift right] \arrow[dl, shift left] \arrow[dr, shift right] \arrow[dr, shift left] \arrow[dr, phantom, "\cong"] & & \\
 & & \mathrm{D}/\mathrm{F}\mathbf{A} \arrow[dl, shift right, "\mathrm{F}(-)"'] \arrow[dl, shift left] \arrow[dr, shift right] \arrow[dr, shift left] \arrow[dr, phantom, "\simeq"] & \\
\mathrm{C}/\mathbf{A} \arrow[rr, shift right, "\mathbf{f}^*"'] \arrow[rr, shift left] & & \mathrm{C}/\mathbf{B} \arrow[ll, shift right, "(\mathrm{F}\mathbf{f})^*"'] \arrow[ll, shift left] \arrow[ur, shift right, "\mathrm{F}(-)"'] \arrow[ur, shift left] & \mathrm{D}/\mathrm{F}\mathbf{B} \arrow[l, shift right] \arrow[l, shift left]
\end{tikzcd}$$

Indexed objects $\mathrm{C} \to \mathcal{P}res\,(\mathcal{B}\,)$

The fulcrum

Definition (Fulcrum)

Given the 2-category $\mathcal{P}res\,(\mathcal{B}\,)$, a **fulcrum** is:

- an object $\mathrm{I} \in \mathcal{B}$
- $\forall \mathrm{E} \in \mathcal{P}res\,(\mathcal{B}\,)$ a 1-morphism $\mathtt{F}_{\mathrm{E}} : \mathrm{I} \to \mathrm{E} \in \mathcal{B}$
- preserved by morphisms of $\mathcal{P}res\,(\mathcal{B}\,)$

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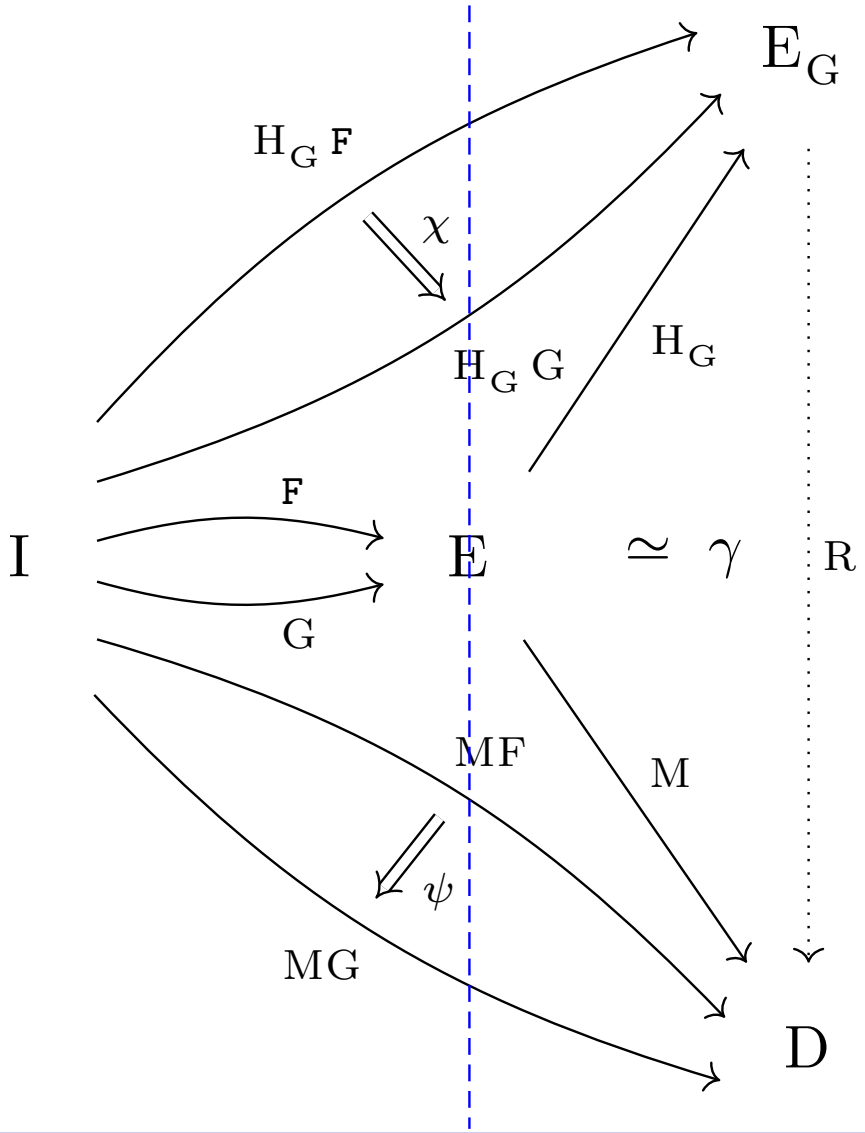
- preserved by morphisms of $\mathcal{P}res\,(\mathcal{B}\,)$

Example

- $\mathcal{C}at$-based $\mathrm{I} = \bullet \circlearrowleft \mathbb{1}$, $\mathtt{F}_{\mathrm{E}} = \mathbf{1}_{\mathrm{E}}$
- Similarly in $\mathcal{P}rod$
- In $(\mathcal{S}ym)\mathcal{M}on\mathcal{C}at$ with strong or colax functors, $\mathrm{I} = \bullet \circlearrowleft \mathbb{1}$ or $\langle \star \rangle$, $\mathtt{F}_{\mathrm{E}} = \mathbf{J}_{\mathrm{E}}$.

The pleronomial object

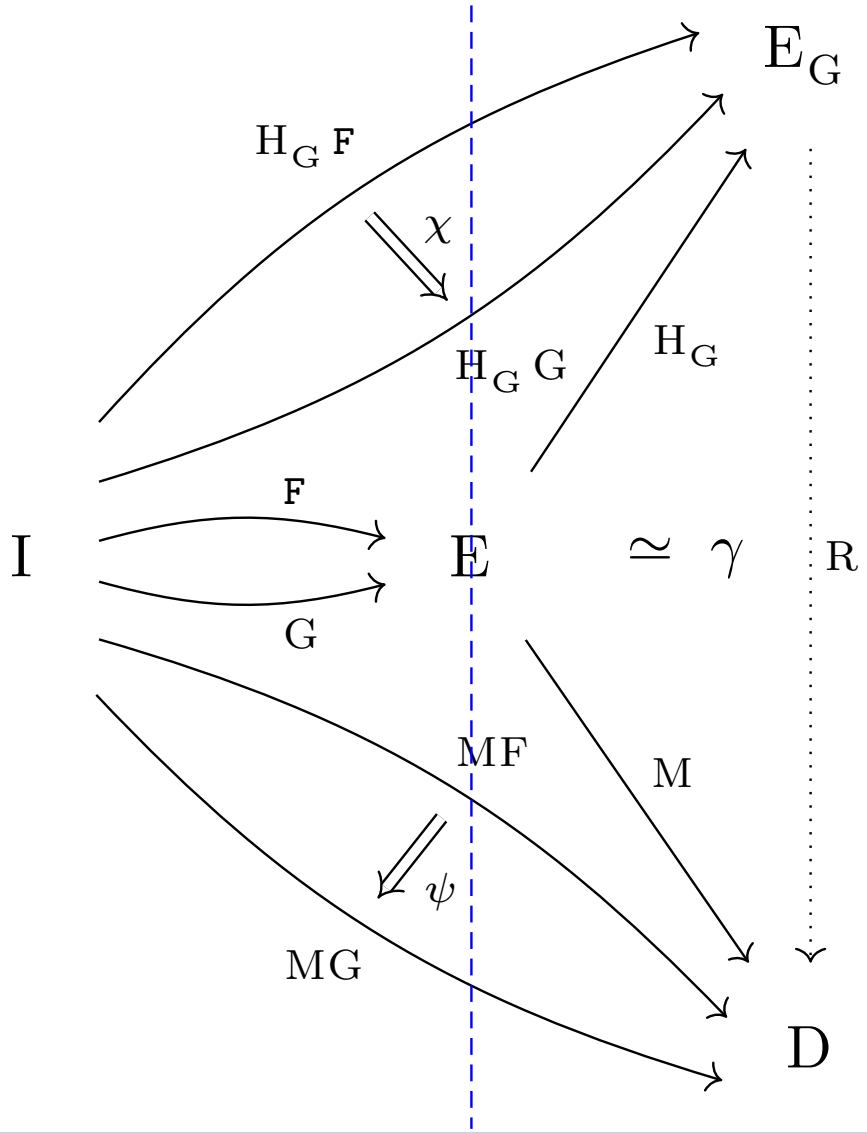
Reformulation in $\mathcal{B}$:
Coinserter with data in $\mathcal{B}$ but universal property in $\mathcal{P}res(\mathcal{B})$.



The pleronomial object

Reformulation in $\mathcal{B}$:
Coinserter with data in $\mathcal{B}$ but universal property in $\mathcal{P}res\,(\mathcal{B}\,)$.
With

- $\mathtt{F}$ the fulcrum
- $\mathrm{I}, \mathrm{G} \in \mathcal{B}$
- $\mathrm{E}_{\mathrm{G}}^{\mathtt{F}}, \mathrm{D}, \mathrm{M}, \mathrm{H}_{\mathrm{G}}^{\mathtt{F}}, \mathrm{R} \in \mathcal{P}res\,(\mathcal{B}\,)$
- $(\gamma\,\mathrm{G}) \bullet (\mathrm{R}\,\chi) \bullet (\gamma^{-1}\,\mathrm{F}) = \psi$.



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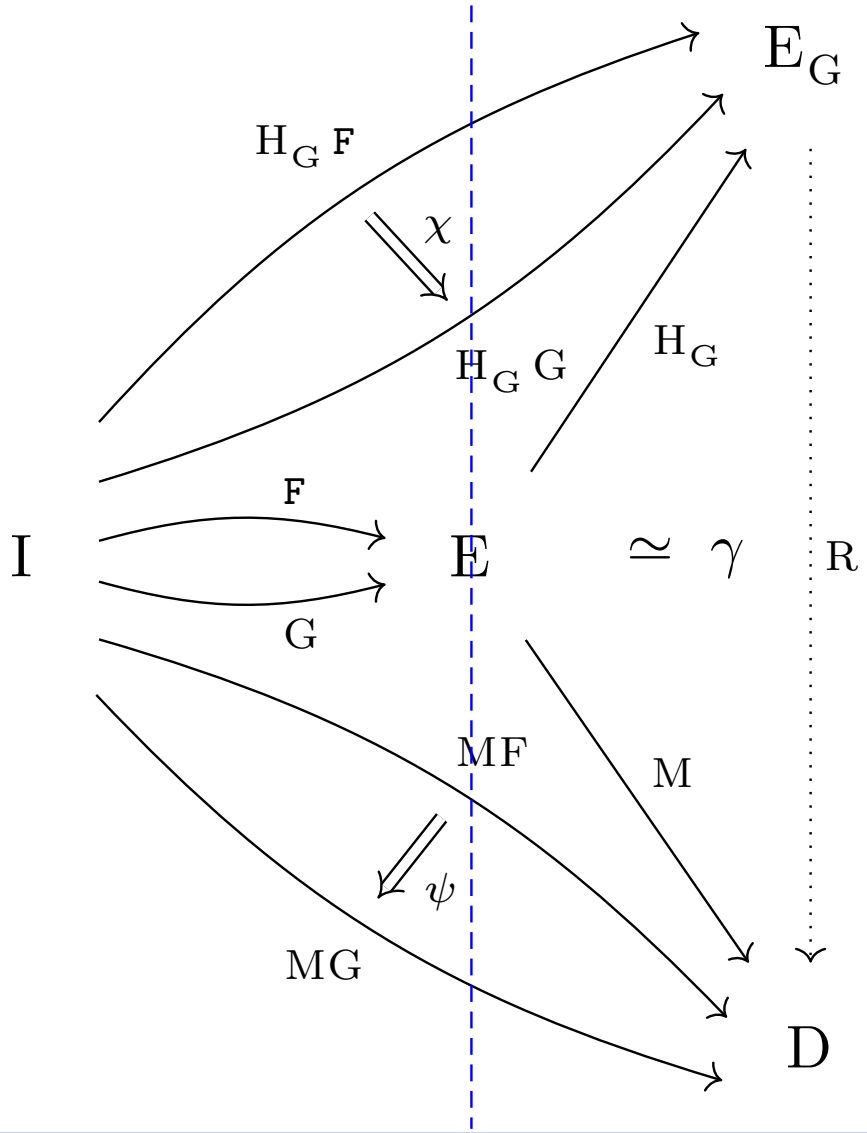
$+$ 2-universal property



Examples

$\mathcal{C}at$	$\mathcal{CAT}$	$\mathcal{P}rod$	$\mathcal{H}o(\mathcal{QC}at)$
small presheaves $\mathcal{P}$	presheaves (if locally small) $\mathcal{P}r$	downward closed sets $\mathcal{D}$	presheaves (if locally small) $\mathcal{P}s$
locally κ-presentable, κ fixed Pleronomial: slice category			κ-presentable, κ fixed Pleronomial: slice ∞-category
reflective subcategories of $\mathcal{P}\,\mathrm{S}$, S small Pleronomial: slice category	total categories that are subobjects of $\mathcal{P}r\,\mathrm{S}$ Pleronomial: slice category	reflective subpreorder of $\mathcal{D}\,\mathrm{P}$ Pleronomial: downset	reflective subquasicategory of $\mathcal{P}s\,\mathrm{S}$ Pleronomial: slice ∞-category
	total categories Pleronomial: slice category	reflective subpreorder of $\mathcal{D}\,\mathrm{P}$ Pleronomial: downset category	

The self-indexing

Proposition

Given a 2-morphism $\alpha : \mathrm{G} \Rightarrow \mathrm{K}$, if the pleronomials exist, there is an induced base-change:

$$\begin{tikzcd}
\mathrm{I} \arrow[r, bend left, "\mathrm{G}"] \arrow[r, bend right, "\mathrm{K}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{E} \arrow[r, "\mathrm{H_G}"] \arrow[dr, "\mathrm{H_K}"'] & \mathrm{E_G} \\
& & \mathrm{E_K} \arrow[u, dotted]
\end{tikzcd}$$

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& & \mathrm{E_K} \arrow[u, dotted]
\end{tikzcd}$$

Theorem

If all pleronomials exist there is a pseudofunctor

$$\begin{aligned}
\mathsf{Self}_{\mathrm{E}} : \quad & \mathcal{B}(\mathrm{I}, \mathrm{E}) && \to \mathcal{P}res(\mathcal{B}) \\
& \mathrm{G} && \mapsto \mathrm{E_G} \\
& \alpha : \mathrm{G} \Rightarrow \mathrm{K} && \mapsto \alpha^*
\end{aligned}$$

and an indexed morphism
$\mathsf{H} : \mathsf{cst}_{\mathrm{E}} \to \mathsf{Self}_{\mathrm{E}}$

Example

In $\mathcal{C}at$, $\mathsf{Self}_{\mathrm{E}} = \mathsf{cod}_{\mathrm{E}}$.

Slicing adjoints

Proposition

Any presentable morphism $\mathrm{R} : \mathrm{D} \to \mathrm{E}$ *induces a 'slicing' presentable morphism* ${}_{\mathrm{G}}\mathrm{R} : \mathrm{D}_{\mathrm{LG}} \to \mathrm{E}_{\mathrm{G}} : {}_{\mathrm{G}}\mathrm{L}$.

$$
\begin{tikzcd}
\mathrm{I} \arrow[r, "\mathrm{G}"'] \arrow[dr, "\mathtt{F}"'] & \mathrm{E} \arrow[r, "\mathrm{H}_{\mathrm{G}}"'] \arrow[d, shift right, "\mathrm{L}"'] \arrow[d, phantom, "\dashv"] & \mathrm{E}_{\mathrm{G}} \arrow[d, shift right, "{}_{\mathrm{G}}\mathrm{L}"'] \arrow[d, phantom, "\dashv"] \\
& \mathrm{D} \arrow[r, "\mathrm{H}_{\mathrm{LG}}"'] \arrow[u, shift right, "\mathrm{R}"'] & \mathrm{D}_{\mathrm{LG}} \arrow[u, shift right, "{}_{\mathrm{G}}\mathrm{R}"']
\end{tikzcd}
$$

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& \mathrm{D} \arrow[r, "\mathrm{H_{LG}}"'] \arrow[u, shift right, "\mathrm{R}"'] & \mathrm{D_{LG}} \arrow[u, shift right, "{}_{\mathrm{G}}\mathrm{R}"']
\end{tikzcd}$$

Example

In $\mathcal{C}at$, these are the classical slicing of adjoint functor.

Topological cosmoi

Slogan

Topos theory = Models + presentability + (base-point + base-change + slicing)

Topological cosmoi

Slogan

Topos theory $=$ Models $+$ presentability $+$ (base-point $+$ base-change $+$ slicing)

Definition

A **topological cosmos** is a 2-category $\mathcal{B}$ equipped with:

- a notion of model
- a sub-2-category $\mathcal{P}res\,(\mathcal{B}\,)$
- a fulcrum

such that every $\mathrm{E} \in \mathcal{P}res\,(\mathcal{B}\,)$ has all pleronomials.

Geometric morphisms

Definition (Geometric morphism)

A presentable $\mathrm{L} \dashv \mathrm{R} : \mathrm{E} \to \mathrm{D} \in \mathcal{P}res\,(\mathcal{B}\,)$ is **geometric** if:

- the $({}_{\mathrm{G}}\mathrm{L}\,)$ form an indexed morphism $\mathsf{L} : \mathsf{Self}_{\mathrm{E}} \to \mathsf{Self}_{\mathrm{LE}}$
- there is an indexed 2-isomorphism $\mathsf{L}\,\mathsf{H}_{\mathrm{E}} \simeq \mathsf{H}_{\mathrm{LE}}\mathrm{L}$

$$\begin{tikzcd}
\mathsf{cst}_{\mathrm{E}} \arrow[r, "\mathsf{cst}_{\mathrm{L}}"] \arrow[d] \arrow[dr, phantom, "\simeq"] & \mathsf{cst}_{\mathrm{D}} \arrow[d] \\
\mathsf{Self}_{\mathrm{E}} \arrow[r, dotted, "\mathsf{L}"'] & \mathsf{Self}_{\mathrm{LE}}
\end{tikzcd}$$

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Unpacking

$$\begin{tikzcd}
\mathrm{E}_{\mathrm{K}} \arrow[r, shift right, "\gamma^*"'] \arrow[d, shift right, "{}_{\mathrm{K}}\mathrm{L}"'] \arrow[r, phantom, "\perp"] & \mathrm{E}_{\mathrm{G}} \arrow[l, shift right] \arrow[d, shift right, "{}_{\mathrm{G}}\mathrm{L}"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{D}_{\mathrm{LK}} \arrow[r, shift right, "(\mathrm{L}\gamma)^*"'] \arrow[r, phantom, "\perp"] \arrow[u, shift right] & \mathrm{D}_{\mathrm{LG}} \arrow[l, shift right] \arrow[u, shift right]
\end{tikzcd}$$

$+$ coherence conditions

Toposes

Definition (Topos)

$\mathrm{E} \in \mathcal{P}res\,(\mathcal{B}\,)$ is a **topos** if there is $\mathrm{L} \dashv \mathrm{R} : \mathrm{E} \to \mathcal{P}\,\mathrm{S}$ geometric such that $\mathrm{LR} \simeq \mathbb{1}_{\mathrm{E}}$.

Slogan

Topos = self-indexing preserving presentable subobject of some model

$$
\begin{tikzcd}
 & & & (\mathcal{P}\,\mathrm{S})_{\mathrm{LG}} \arrow[d, shift right] \\
\mathcal{P}\,\mathrm{S} \arrow[rrr] \arrow[d, shift right] \arrow[dr] & & (\mathcal{P}\,\mathrm{S})_{\mathrm{LK}} \arrow[ur] \arrow[d, shift right] & \\
 & (\mathcal{P}\,\mathrm{S})_{\mathrm{LN}} \arrow[ur] \arrow[d, shift right] & & \mathrm{E}_{\mathrm{G}} \arrow[uu, shift right] \\
\mathrm{E} \arrow[uu, shift left] \arrow[rrr] \arrow[dr] \arrow[urrr] & & \mathrm{E}_{\mathrm{K}} \arrow[uu, shift left] \arrow[ur] & \\
 & \mathrm{E}_{\mathrm{N}} \arrow[uu, shift left] \arrow[ur] & &
\end{tikzcd}
$$

Examples

$\mathcal{C}at$	$\mathcal{CAT}$	$\mathcal{P}rod$	$\mathcal{H}o(\mathcal{QC}at)$
small presheaves $\mathcal{P}$	presheaves (if locally small) $\mathcal{P}r$	downward closed sets $\mathcal{D}$	presheaves (if locally small) $\mathcal{P}s$
locally κ-presentable, κ fixed Pleronomial: slice category locally κ-presentable toposes			κ-presentable, κ fixed Pleronomial: slice ∞-category Toposes: κ-presentable ∞-toposes
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Alternative characterisations

Geometric monads

Hypothesis

(EM) $\mathcal{B}$ has Eilenberg-Moore objects of idempotent monads

Proposition

Under (EM), toposes are equivalently the Eilenberg-Moore objects of geometric idempotent monad on some model $\mathcal{P}\,\mathrm{S}$.

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Proposition

Under (EM), toposes are equivalently the Eilenberg-Moore objects of <u>*geometric*</u> *idempotent monad on some model* $\mathcal{P}\,\mathrm{S}$.

Universal monad (closure) operators

Hypothesis

(IH) ${}_{\mathrm{G}}\mathrm{L}\,{}_{\mathrm{G}}\mathrm{R} \simeq \mathbb{1}$ *whenever* $\mathrm{LR} \simeq \mathbb{1}$

(T) $\mathcal{B}\,(\mathrm{I}, \mathrm{E})$ *has a terminal object*

Proposition

Under (IH) and (T), toposes are equivalently given by indexed idempotent monads $\mathsf{M} = (\mathrm{M}_{\mathrm{G}})$ *on* $\mathsf{Self}_{\mathrm{E}}$.

Monoidal perspectives

e.g. (Borceux and Bossche 1986; Borceux and Cruciani 1993; Heymans 2010; Tenório, Mendes, and Mariano 2022),...

Setting

The 2-category: $\mathcal{S}ym\mathcal{M}on\mathcal{C}at_c$
The models: Day convolution
The presentable objects
'Promonoidally' locally presentable categories
The points: $\langle\star\rangle \to \mathrm{E}$ corresponding to $(\mathbf{J}, \mathbf{A})$
The fulcrum: $\langle\star\rangle$, picking the unit

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Theorem

Suppose $\mathbf{C}$ *is presentable and (TPC)* $\otimes$ *preserves coreflexive equalisers then:*

- C *has cofree comonoids* $!\mathbf{A}, \mathbf{A} \in \mathrm{C}$
- $\mathrm{C}_{\mathbf{A}}$ *for* $\mathbf{A} \in \mathrm{C}$ *is* $!\mathbf{A} - \mathrm{Comod}$
- *base-change is extension of scalars*

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- *base-change is extension of scalars*

Example

Always true in $\mathcal{P}rod$. Toposes are the posets whose reflector preserves $-\&!\mathbf{A}$.

Example

True for cartesian categories, pleronomials are again the slices.

Slogans

Topos theory = Models + presentability + (base-point + base-change + slicing)
Topos = self-indexing preserving presentable subobject of some model

Conjecture (Refining the theory)

The following are toposes (with weak models):

- *∞-pretoposes*
- *Regular categories*
- *Barr-exact categories*
- *Lextensive categories*
- *...*

Conjecture (Cosmology)

There are change of cosmoi operations mirroring those in enriched category. Localic toposes arise from such an operation.

Conjecture (Giraud's theorem)

If $\mathsf{Self}_{\mathcal{P}\,\mathrm{S}}$ *is closed,* E *is a topos if and only if*

- *it is presentable*
- $\mathsf{Self}_{\mathrm{E}}$ *is closed*
- $\mathsf{Self}_{\mathrm{E}}$ *satisfies descent whenever all* $\mathsf{Self}_{\mathcal{P}\,\mathrm{S}}$

Conjecture (Logic and type theory)

There is an extension of comprehension categories through the self-indexing.

$$\begin{tikzcd} \mathrm{D} \arrow[rr] \arrow[dr] & & \vec{\mathrm{C}} \arrow[dl] \\ & \mathrm{C} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{D} \arrow[d] \\ \mathsf{Self}_{\mathrm{E}} \end{tikzcd}$$

Formal category theory

Formal category theory = category theory in a 2-category $\mathcal{B}$
↳Category theory = Formal category theory in $\mathcal{C}at$

category $\mathrm{C} \in \mathcal{C}at$ $\leftrightsquigarrow$ object $\mathrm{E} \in \mathcal{B}$

object $\mathbf{A} \in \mathrm{C}$ $\leftrightsquigarrow$ 1-morphism $\mathrm{I} \to \mathrm{E}$

morphism $\mathbf{f} : \mathbf{A} \to \mathbf{B} \in \mathrm{C}$ $\leftrightsquigarrow$ triangle

$$\begin{tikzcd} \mathrm{I} \arrow[dd] \arrow[dr] & \\ & \mathrm{E} \\ \mathrm{J} \arrow[ur] & \end{tikzcd} \quad \Downarrow$$

Easy

- Adjunctions
- Monads

Hard(-ish)

- (Co)limits
- Presheaves

The elephant (Johnstone 2002)

1. A topos is a category of (pre)sheaves on a site (= pullback-stable).
2. A topos is a left-exact reflective subcategory of the category of presheaves of a small category.
3. A topos is the category of algebras of a left-exact idempotent monad on the category of presheaves of a small category.
4. A topos is the category of sheaves for a Lawvere-Tierney topology(= pullback-stable) on the category of presheaves of a small category.
5. A topos is the localisation of the category of presheaves of a small category at a class of morphisms satisfying the right Ore conditions
6. A topos is locally presentable category satisfying the Giraud conditions
7. A topos is lex total category with a small set of generators.

2-category	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{C}at$	$\mathcal{CAT}$	$\mathcal{P}rod$	$\mathcal{H}o(\mathcal{QC}at)$
Models	KZ-monad $\mathcal{M}$ (Kock 1995; Zöberlein 1976)	Yoneda structure $\mathcal{Y}$ (Street and Walters 1978)	small presheaves $\mathcal{P}$	presheaves (if locally small) $\mathcal{P}r$	downward closed sets $\mathcal{D}$	presheaves $\mathcal{P}s$
strong size conditions	KZ-context (Di Liberti and Loregian 2023)		locally κ-presentable, κ fixed Pleronomial: slice category locally κ-presentable toposes			κ-presentable, κ fixed Pleronomial: slice ∞-category Toposes: κ-presentable ∞-toposes
mild size conditions	cocomplete object that are subobjects of $\mathcal{M}\,\mathrm{S}$, S $\mathcal{M}$-small	total objects that are subobjects of $\mathcal{Y}\,\mathrm{S}$, S $\mathcal{Y}$-small	reflective subcategories of $\mathcal{P}\,\mathrm{S}$, S small Pleronomial: slice category Toposes: toposes	total categories that are subobjects of $\mathcal{P}r\,\mathrm{S}$ Pleronomial: slice category Toposes: toposes	reflective subpreorder of $\mathcal{D}\,\mathrm{P}$ Pleronomial: downset Toposes: locales	reflective subquasicategory of $\mathcal{P}s\,\mathrm{S}$ Pleronomial: slice ∞-category Toposes: $\supseteq$ ∞-toposes
no size conditions	cocomplete object + right adjoints	total objects		total categories Pleronomial: slice category Toposes: lex total categories	reflective subpreorder of $\mathcal{D}\,\mathrm{P}$ Pleronomial: downset category Toposes: locales	

Left-exactness from base-change

Slogan

Topos theory = Models + presentability + (base-point + base-change + slicing)

$$\begin{tikzcd}
 & \mathrm{cst}_{\mathrm{D}} \arrow[d] \\
\mathrm{cst}_{\mathrm{C}} \arrow[ur, "\mathrm{cst}_{\mathrm{F}}"] \arrow[d] \arrow[dr, phantom, "\simeq"] & \\
\mathrm{cod}_{\mathrm{C}} \arrow[r, dotted, "\mathrm{F}"] & \mathrm{cod}_{\mathrm{FC}}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathrm{D} \arrow[dl, shift right, "\mathrm{F}"'] \arrow[dl, shift left] \arrow[dd, shift right] \arrow[dd, shift left] \arrow[dr, shift right] \arrow[dr, shift left] & \\
 & \mathrm{C} \arrow[ur, shift right] \arrow[ur, shift left] \arrow[dl, shift right] \arrow[dl, shift left] \arrow[dr, shift right] \arrow[dr, shift left] \arrow[dr, phantom, "\cong"] & & \\
 & & \mathrm{D}/\mathrm{F}\mathbf{A} \arrow[dl, shift right, "\mathrm{F}(-)"'] \arrow[dl, shift left] \arrow[dr, shift right, "\simeq"'] & \\
\mathrm{C}/\mathbf{A} \arrow[rr, shift right, "\mathbf{f}^*"'] \arrow[rr, shift left, dotted, "\mathrm{F}(-)"] & & \mathrm{C}/\mathbf{B} \arrow[ll, shift right] \arrow[ll, shift left, dotted, "\mathrm{F}(-)"] & \mathrm{D}/\mathrm{F}\mathbf{B} \arrow[l, shift right, "(\mathrm{F}\mathbf{f})^*"'] \arrow[l, shift left]
\end{tikzcd}$$

Indexed objects $\mathrm{C} \to \mathcal{P}res\,(\mathcal{B}\,)$

Self-indexing

Q: How to generalise the codomain fibration ?
A: With a universal property of slices...but which?

Not with commas

- enriched commas are augmented $\to$comma at terminal object is not the category itself.
- higher slice is not (but almost) a comma
- for monoidal categories and colax functors, the comma is $\sqcup_{n\in\mathbb{N}}\mathrm{C}/(\mathbf{A}^{\otimes n})$

Polynomial objects

Theorem (Artin, Grothendieck, and Verdier 1971; Lambek and Scott 1988; Hermida and Jacobs 1995)

Let C *be a* ~~*finitely complete category*~~ *presentable category*, $\mathbf{A} \in \mathrm{C}$ *then* $\mathrm{C}/\mathbf{A}$ *satisfies the following universal property:*

$$\begin{tikzcd}
\mathrm{C} \arrow[rr, "\mathrm{H}"'] \arrow[dr, "\mathrm{M}"'] \arrow[rr, phantom, shift left=3ex, "\perp"] & & \mathrm{C}/\mathbf{A} \arrow[dl, dotted, "\exists!"'] \arrow[ll, bend right, "\vdash"'] \\
& \mathrm{D} \text{ presentable} \arrow[ul, bend left, "\dashv"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\chi : \mathrm{H}\mathbf{1} \arrow[d, mapsto] \\
\psi : \mathrm{M}\mathbf{1} \arrow[r, phantom, "\rightarrow"] & \mathrm{M}\mathbf{A}
\end{tikzcd}$$

Motivation: The same theory but with an extra constant/variable of sort/type $\mathbf{A}$ *.*

The fulcrum

Definition (Fulcrum)

Given the 2-category $\mathcal{P}res\,(\mathcal{B}\,)$, a **fulcrum** is:

- an object $\mathrm{I} \in \mathcal{B}$
- $\forall \mathrm{E} \in \mathcal{P}res\,(\mathcal{B}\,)$ a 1-morphism $\mathtt{F}_{\mathrm{E}} : \mathrm{I} \to \mathrm{E} \in \mathcal{B}$
- preserved by morphisms of $\mathcal{P}res\,(\mathcal{B}\,)$

Example

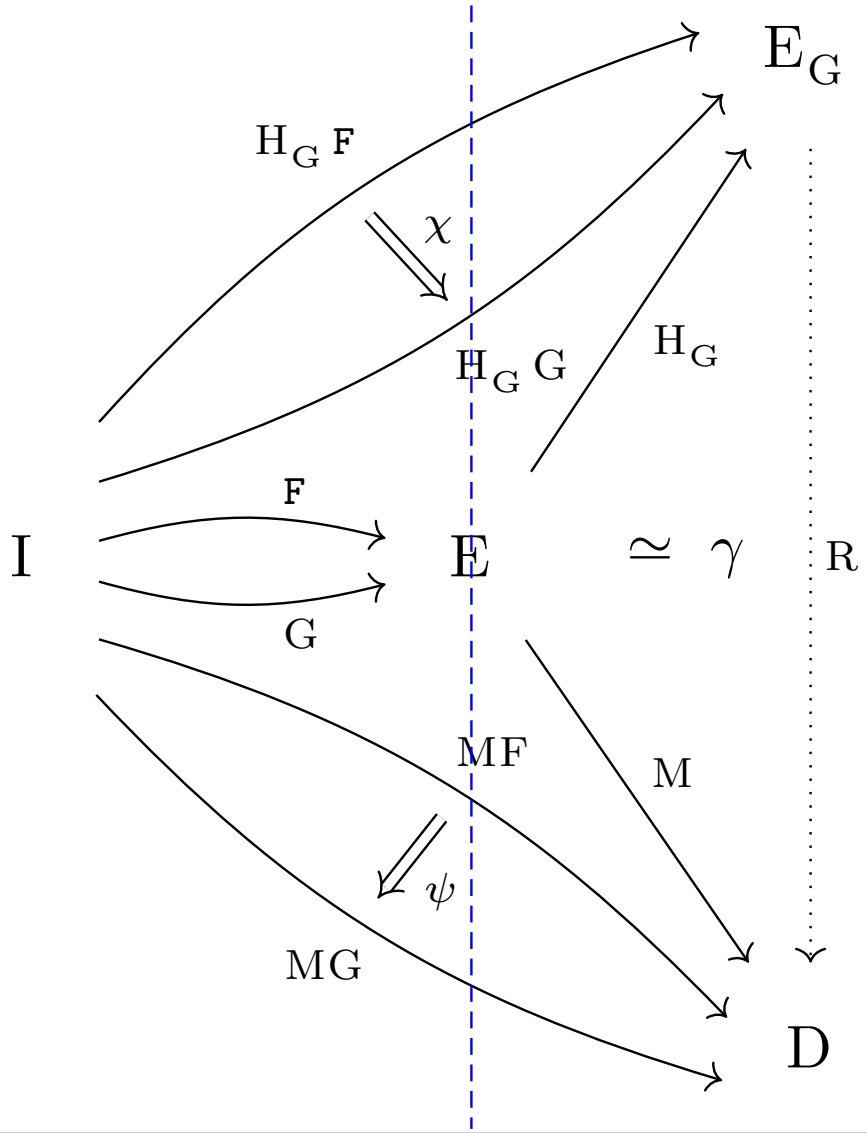
- $\mathcal{C}at$-based $\mathrm{I} = \bullet \circlearrowleft \mathbb{1}$, $\mathtt{F}_{\mathrm{E}} = \mathbf{1}_{\mathrm{E}}$
- Similarly in $\mathcal{P}rod$
- In $(\mathcal{S}ym)\mathcal{M}on\mathcal{C}at$ with strong or colax functors, $\mathrm{I} = \bullet \circlearrowleft \mathbb{1}$ or $\langle \star \rangle$, $\mathtt{F}_{\mathrm{E}} = \mathbf{J}_{\mathrm{E}}$.

The pleronomial object

Reformulation in $\mathcal{B}$:
Coinserter with data in $\mathcal{B}$ but universal property in $\mathcal{P}res\,(\mathcal{B}\,)$.
With

- $\mathtt{F}$ the fulcrum
- $\mathrm{I}, \mathrm{G} \in \mathcal{B}$
- $\mathrm{E}_{\mathrm{G}}^{\mathtt{F}}, \mathrm{D}, \mathrm{M}, \mathrm{H}_{\mathrm{G}}^{\mathtt{F}}, \mathrm{R} \in \mathcal{P}res\,(\mathcal{B}\,)$
- $(\gamma\,\mathrm{G}) \bullet (\mathrm{R}\,\chi) \bullet (\gamma^{-1}\,\mathrm{F}) = \psi$.

$+$ 2-universal property



The self-indexing

Proposition

Given a 2-morphism $\alpha : \mathrm{G} \Rightarrow \mathrm{K}$, if the pleronomials exist, there is an induced base-change:

$$\begin{tikzcd}
\mathrm{I} \arrow[r, bend left, "\mathrm{G}"] \arrow[r, bend right, "\mathrm{K}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{E} \arrow[r, "\mathrm{H_G}"] \arrow[dr, "\mathrm{H_K}"'] & \mathrm{E_G} \\
& & \mathrm{E_K} \arrow[u, dotted]
\end{tikzcd}$$

Theorem

If all pleronomials exist there is a pseudofunctor

$$\begin{aligned}
\mathsf{Self}_{\mathrm{E}} : \quad & \mathcal{B}(\mathrm{I}, \mathrm{E}) && \to \mathcal{P}res(\mathcal{B}) \\
& \mathrm{G} && \mapsto \mathrm{E_G} \\
& \alpha : \mathrm{G} \Rightarrow \mathrm{K} && \mapsto \alpha^*
\end{aligned}$$

and an indexed morphism
$\mathsf{H} : \mathsf{cst}_{\mathrm{E}} \to \mathsf{Self}_{\mathrm{E}}$

Example

In $\mathcal{C}at$, $\mathsf{Self}_{\mathrm{E}} = \mathsf{cod}_{\mathrm{E}}$.

Slicing adjoints

Proposition

Any presentable morphism $\mathrm{R} : \mathrm{D} \to \mathrm{E}$ *induces a 'slicing' presentable morphism* ${}_{\mathrm{G}}\mathrm{R} : \mathrm{D}_{\mathrm{LG}} \to \mathrm{E}_{\mathrm{G}} : {}_{\mathrm{G}}\mathrm{L}$.

$$\begin{tikzcd}
\mathrm{I} \arrow[r, "\mathrm{G}"'] \arrow[dr, "\mathtt{F}"'] & \mathrm{E} \arrow[r, "\mathrm{H_G}"'] \arrow[d, shift right, "\mathrm{L}"'] \arrow[dr, phantom, "\dashv"] & \mathrm{E_G} \arrow[d, shift right, "{}_{\mathrm{G}}\mathrm{L}"'] \arrow[dr, phantom, "\dashv"] & \\
& \mathrm{D} \arrow[r, "\mathrm{H_{LG}}"'] \arrow[u, shift right, "\mathrm{R}"'] & \mathrm{D_{LG}} \arrow[u, shift right, "{}_{\mathrm{G}}\mathrm{R}"'] &
\end{tikzcd}$$

Example

In $\mathcal{C}at$, these are the classical slicing of adjoint functor.

Topological cosmoi

Slogan

Topos theory $=$ Models $+$ presentability $+$ (base-point $+$ base-change $+$ slicing)

Definition

A **topological cosmos** is a 2-category $\mathcal{B}$ equipped with:

- a notion of model
- a sub-2-category $\mathcal{P}res\,(\mathcal{B}\,)$
- a fulcrum

such that every $\mathrm{E} \in \mathcal{P}res\,(\mathcal{B}\,)$ has all pleronomials.

Geometric morphisms

Definition (Geometric morphism)

A presentable $\mathrm{L} \dashv \mathrm{R} : \mathrm{E} \to \mathrm{D} \in \mathcal{P}res\,(\mathcal{B}\,)$ is **geometric** if:

- the $({}_{\mathrm{G}}\mathrm{L}\,)$ form an indexed morphism $\mathsf{L} : \mathsf{Self}_{\mathrm{E}} \to \mathsf{Self}_{\mathrm{LE}}$
- there is an indexed 2-isomorphism $\mathsf{L}\,\mathsf{H}_{\mathrm{E}} \simeq \mathsf{H}_{\mathrm{LE}}\mathrm{L}$

Unpacking

$$\begin{tikzcd}
\mathrm{E}_{\mathrm{K}} \arrow[r, shift right, "\gamma^*"'] \arrow[d, shift right, "{}_{\mathrm{K}}\mathrm{L}"'] \arrow[r, phantom, "\perp"] & \mathrm{E}_{\mathrm{G}} \arrow[l, shift right] \arrow[d, shift right, "{}_{\mathrm{G}}\mathrm{L}"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{D}_{\mathrm{LK}} \arrow[r, shift right, "(\mathrm{L}\gamma)^*"'] \arrow[r, phantom, "\perp"] \arrow[u, shift right] & \mathrm{D}_{\mathrm{LG}} \arrow[l, shift right] \arrow[u, shift right] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

$+$ coherence conditions

Toposes

Definition (Topos)

$\mathrm{E} \in \mathcal{P}res\,(\mathcal{B}\,)$ is a **topos** if there is $\mathrm{L} \dashv \mathrm{R} : \mathrm{E} \to \mathcal{P}\,\mathrm{S}$ geometric such that $\mathrm{LR} \simeq \mathbb{1}_{\mathrm{E}}$.

Slogan

Topos = self-indexing preserving presentable subobject of some model

$$
\begin{tikzcd}
 & & & (\mathcal{P}\,\mathrm{S})_{\mathrm{LG}} \arrow[d, shift right] \\
\mathcal{P}\,\mathrm{S} \arrow[rrr] \arrow[d, shift right] \arrow[dr] & & (\mathcal{P}\,\mathrm{S})_{\mathrm{LK}} \arrow[ur] \arrow[d, shift right] & \\
 & (\mathcal{P}\,\mathrm{S})_{\mathrm{LN}} \arrow[ur] \arrow[d, shift right] & & \mathrm{E}_{\mathrm{G}} \arrow[uu, shift right] \\
\mathrm{E} \arrow[uu, shift left] \arrow[rrr] \arrow[dr] \arrow[urrr] & & \mathrm{E}_{\mathrm{K}} \arrow[ur] \arrow[uu, shift left] & \\
 & \mathrm{E}_{\mathrm{N}} \arrow[ur] \arrow[uu, shift left] & &
\end{tikzcd}
$$

Alternative characterisations

Geometric monads

Hypothesis

(EM) $\mathcal{B}$ *has Eilenberg-Moore objects of idempotent monads*

Proposition

Under (EM), toposes are equivalently the Eilenberg-Moore objects of <u>*geometric*</u> *idempotent monad on some model* $\mathcal{P}\,\mathrm{S}$.

Universal monad (closure) operators

Hypothesis

(IH) ${}_{\mathrm{G}}\mathrm{L}\,{}_{\mathrm{G}}\mathrm{R} \simeq \mathbb{1}$ *whenever* $\mathrm{LR} \simeq \mathbb{1}$

(T) $\mathcal{B}\,(\mathrm{I}, \mathrm{E})$ *has a terminal object*

Proposition

Under (IH) and (T), toposes are equivalently given by indexed idempotent monads $\mathsf{M} = (\mathrm{M}_{\mathrm{G}})$ *on* $\mathsf{Self}_{\mathrm{E}}$.

Future works

- Refining the theory
  - ↳ Adjoint free formulation by focusing on the left adjoint
    →working with 'lex' objects, locally close objects
  - ↳ Do away with the fulcrum
- More topos theory
  - ↳ Giraud's theorem and other characterisation
  - ↳ Fundamental theorem
  - ↳ ...
- Cosmology
- Connection with logic
- Refining the monoidal theory
  - ↳ general case
  - ↳ applications
  - ↳ linear logic
- Enriched case
- (n, r) toposes

Finding pleronomials

$$\begin{tikzcd}
\mathrm{I} \arrow[r, "\mathrm{G}"] & \mathrm{E} \arrow[r, shift left] \arrow[dr, shift left, "\vdash"] \arrow[dr, phantom, shift right=3, "\perp"] & \mathrm{E}_{\mathrm{G}} \arrow[l, shift left] \arrow[dl, shift left, "\mathrm{N}"] \\
& & \mathrm{D}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{D} \arrow[d, "\sim"] \\
\mathrm{D}_{\mathrm{G}_0} \arrow[d, "{}_{\mathrm{G}_0}\mathrm{N}"] \\
\mathrm{E}_{\mathrm{NG}_0} \arrow[d, "\alpha^*"] \\
\mathrm{E}_{\mathrm{G}}
\end{tikzcd}$$

Slogan

Topos theory = Models + presentability + (base-point + base-change + slicing)

Proof strategy

(Facultative:) S_0 Base object — Property of the base object

$$\downarrow$$

Models $\mathcal{P}\,\mathrm{S}(= [\mathrm{S}^{op}, \mathrm{S}_0])$ — Property for models

$$\downarrow$$

Toposes $\mathrm{E} \underset{\hookrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{P}\,\mathrm{S}$ — Property for toposes through the self-indexing

Choice of structure

The 2-category: Symmetric monoidal categories with colax functors (closest to cartesian case)

The models: Presheaves over small categories with Day convolution

The fulcrum: $\mathrm{I} = \langle \star \rangle$, picking the unit

Points: Functors $\mathrm{I} \to \mathrm{E}$ corresponding to $(\mathbf{J}, \mathbf{A})$, $\mathbf{J}$ unit of E, $\mathbf{A} \in \mathrm{E}$

Recall

- Doctrinal adjunction $\to$right adjoints are strong

Monoidal presentability

Definition

Promonoidal category = monoidal category with structure without representatives i.e.

- profunctor $\mathbb{P} : \mathrm{C} \times \mathrm{C} \times \mathrm{C}^{op} \to \mathrm{Sets} \leftrightsquigarrow \mathrm{C}(-, - \otimes -)$
- presheaf $\mathbb{J} : \mathrm{C}^{op} \to \mathrm{Sets} \leftrightsquigarrow \mathrm{C}(-, \mathbf{J})$

Definition

C monoidal category is κ-monoidally presentable if:

- there is a colax adjunction $\mathrm{R} : \mathrm{C} \leftrightarrows \mathcal{P}\,\mathrm{S} : \mathrm{L}$, S small promonoidal
- there is T small promonoidal such that $\mathrm{C} = \mathcal{I}nd_{\kappa}\mathrm{T}$

The pleronomial object

Idea: Guess the left adjoint

In $\mathcal{C}at$, given adjoints $\mathrm{M} : \mathrm{E} \to \mathrm{D} : \mathrm{N}$ and $\chi : \mathrm{M}\mathbf{1}_{\mathrm{E}} \to \mathrm{M}\mathbf{A} \leftrightsquigarrow \overline{\chi} : \mathrm{NM}\mathbf{1} \to \mathbf{A}$. For $\mathbf{B} \in \mathrm{D}$

$$\mathbf{B} \xrightarrow{!_{\mathbf{B}}} \mathbf{1}_{\mathrm{D}}$$

The pleronomial object

Idea: Guess the left adjoint

In $\mathcal{C}at$, given adjoints $\mathrm{M} : \mathrm{E} \to \mathrm{D} : \mathrm{N}$ and $\chi : \mathrm{M}\mathbf{1}_{\mathrm{E}} \to \mathrm{M}\mathbf{A} \leftrightsquigarrow \overline{\chi} : \mathrm{N}\mathrm{M}\mathbf{1} \to \mathbf{A}$. For $\mathbf{B} \in \mathrm{D}$

$$\mathrm{N}\mathbf{B} \xrightarrow{\mathrm{N}!_{\mathbf{B}}} \mathrm{N}\mathbf{1}_{\mathrm{D}}$$

The pleronomial object

Idea: Guess the left adjoint

In $\mathcal{C}at$, given adjoints $\mathrm{M} : \mathrm{E} \to \mathrm{D} : \mathrm{N}$ and $\chi : \mathrm{M}\mathbf{1}_{\mathrm{E}} \to \mathrm{M}\mathbf{A} \leftrightsquigarrow \overline{\chi} : \mathrm{NM}\mathbf{1} \to \mathbf{A}$. For $\mathbf{B} \in \mathrm{D}$

$$\mathrm{N}\mathbf{B} \xrightarrow{\mathrm{N}!_{\mathbf{B}}} \mathrm{N}\mathbf{1}_{\mathrm{D}} \xrightarrow{\simeq} \mathrm{NM}\mathbf{1}_{\mathrm{E}} \xrightarrow{\overline{\chi}} \mathbf{A}$$

The pleronomial object

Idea: Guess the left adjoint

In $\mathcal{S}ym\mathcal{M}on\mathcal{C}at$, given $\mathrm{M} : \mathrm{E} \to \mathrm{D} : \mathrm{N}$ and $\chi : \mathrm{M}\mathbf{J} \to \mathrm{M}\mathbf{A} \leftrightsquigarrow \overline{\chi} : \mathrm{NM}\mathbf{J} \to \mathbf{A}$. For $\mathbf{B} \in \mathrm{D}$

$$\mathrm{N}\mathbf{B} \xrightarrow{\mathrm{N}!_{\mathbf{B}}} \mathrm{N}\mathbf{1}_{\mathrm{D}} \xrightarrow{\simeq} \mathrm{NM}\mathbf{1}_{\mathrm{E}} \xrightarrow{\overline{\chi}} \mathbf{A}$$

$$\begin{tikzcd}
\mathrm{N}\mathbf{B} \arrow[d, "\mathrm{N}\lambda_{\mathbf{B}}"] & & \mathrm{N}\mathbf{B} \otimes_{\mathrm{E}} \mathbf{A} \\
\mathrm{N}(\mathbf{B} \otimes_{\mathrm{D}} \mathbf{J}) \arrow[r, "\delta_{\mathrm{B}\otimes\mathbf{J}}"] & \mathrm{N}\mathbf{B} \otimes_{\mathrm{E}} \mathrm{N}\mathbf{J} \arrow[r, "\simeq"] & \mathrm{N}\mathbf{B} \otimes_{\mathrm{E}} \mathrm{NM}\mathbf{J}_{\mathrm{E}} \arrow[u, "\mathbb{1}_{\mathrm{N}\mathbf{B}} \otimes_{\mathrm{E}} \overline{\chi}"']
\end{tikzcd}$$

The pleronomial object

Idea: Guess the left adjoint

In $\mathcal{S}ym\mathcal{M}on\mathcal{C}at$, given $\mathrm{M} : \mathrm{E} \to \mathrm{D} : \mathrm{N}$ and $\chi : \mathrm{M}\mathbf{J} \to \mathrm{M}\mathbf{A} \leftrightsquigarrow \overline{\chi} : \mathrm{NM}\mathbf{J} \to \mathbf{A}$. For $\mathbf{B} \in \mathrm{D}$

$$\mathrm{N}\mathbf{B} \xrightarrow{\mathrm{N}!_{\mathbf{B}}} \mathrm{N}\mathbf{1}_{\mathrm{D}} \xrightarrow{\simeq} \mathrm{NM}\mathbf{1}_{\mathrm{E}} \xrightarrow{\overline{\chi}} \mathbf{A}$$

$$\begin{tikzcd}
\mathrm{N}\mathbf{B} \arrow[d, "\mathrm{N}\lambda_{\mathbf{B}}"] & & \mathrm{N}\mathbf{B} \otimes_{\mathrm{E}} \mathbf{A} \\
\mathrm{N}(\mathbf{B} \otimes_{\mathrm{D}} \mathbf{J}) \arrow[r, "\delta_{\mathrm{B}\otimes\mathbf{J}}"] & \mathrm{N}\mathbf{B} \otimes_{\mathrm{E}} \mathrm{N}\mathbf{J} \arrow[r, "\simeq"] & \mathrm{N}\mathbf{B} \otimes_{\mathrm{E}} \mathrm{NM}\mathbf{J}_{\mathrm{E}} \arrow[u, "\mathbb{1}_{\mathrm{N}\mathbf{B}} \otimes_{\mathrm{E}} \overline{\chi}"']
\end{tikzcd}$$

But $\mathrm{NM}\mathbf{J}_{\mathrm{E}}$ is a comonoid and $\mathrm{N}\mathbf{B}$ a comodule over it.

The pleronomial object II

Theorem

Suppose $\mathbf{C}$ *is presentable and* $\otimes$ *preserves coreflexive equalisers in each argument then:*

- C *has cofree comonoids* $!\mathbf{A}, \mathbf{A} \in \mathrm{C}$
- $\mathrm{C}_{\mathbf{A}}$ *for* $\mathbf{A} \in \mathrm{C}$ *is* $!\mathbf{A} - \mathrm{Comod}$
- *base-change is extension of scalars*

Example

Always true in $\mathcal{P}rod$. Toposes are the posets whose reflector preserves $-\&!\mathbf{A}$.

Example

True for cartesian categories, pleronomials are again the slices.