

A TOPOS FOR ÉTALE-FINITE HEYTING ALGEBRAS

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Lattices of truth-values in toposes

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Elementary toposes and Heyting algebras

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2. In **Set**[→], it is **3** = $(0 < a < 1)$;
3. In **Sh**($\mathcal{O}(X)$), it is $\mathcal{O}(X)$;
4. In **Eff** it is $\mathcal{P}(\mathbb{N}) / \sim$, the “realizability locale”.

All lattices of truth-values are *Heyting algebras*:

Definition

An algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ is called a **Heyting algebra** if:

1. $(H, \wedge, \vee, 0, 1)$ is a (distributive) lattice.
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Those algebras which appear in this way we call **topos-admissible**.

For L a locale, we have:

$$\mathrm{Sub}_{\mathrm{Sh}(L)}(1) \cong L.$$

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Theorem (Freyd (197?))

Every Boolean algebra is topos-admissible.

So we have the following **concrete problem** (possibly originally due to Freyd, popularized by Pitts and Pataia):

Question

Is every Heyting algebra topos-admissible?

We briefly interrupt the regularly
scheduled broadcast

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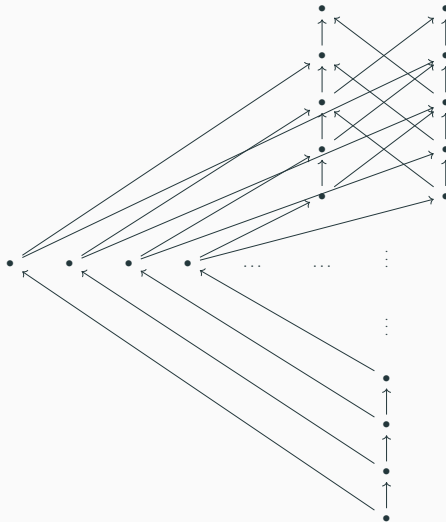
High-level idea: in third-order logic it is possible to encode the existence of definable infinite joins, as least-fixed points of a particular countable approximation.

The same idea with some small refinements can be adapted to many other algebras:

Not all Heyting algebras: example

Theorem (Abbadini, A., Arrieta, 2026)

*The algebra of finite and cofinite upsets of the **Fine poset** is not topos-admissible.*



Which Heyting algebras?

We can reformulate the question to ask for a [characterization](#):

Question

Which Heyting algebras are topos-admissible?

From above we have that **complete Heyting algebras** and **Boolean algebras** are **topos-admissible**. Products, homomorphic images, and special gluings of these, are all also topos-admissible.

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This **properly extended** the class of Heyting algebras previously known to be topos-admissible.

We have in the mean time been able to obtain the following broader result:

Theorem (Abbadini, A., Arrieta, 2026)

Every **locally-finite** Heyting algebra is topos-admissible.

However, the construction is less transparent; for today, we will be focusing on a more particular result concerning **sheaves**.

Étale-finite Heyting algebras

We work with a class introduced by E. Kuznetsov (2024):

Definition (Étale-finite Heyting algebra)

A Heyting algebra H is **étale-finite** if there is a finite Heyting algebra H_0 and a Heyting algebra homomorphism $f: H_0 \rightarrow H$ satisfying **Jibladze's law**: for every $x \in H$:

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Conceptually: there are **finitely many definable truth-values**.

Example

Finite algebras are étale-finite. For subdirectly irreducible, étale-finiteness is just finiteness.

We also work with their **duals**:

Definition (Esakia space)

An ordered space $(X, \leq, \tau)$ is an **Esakia space** if:

1. (X, τ) is compact;
2. (Priestley) If $x \not\leq y$ there is a clopen upset U such that $x \in U$ and $y \notin U$;
3. (Esakia) $\downarrow W = \{x \in X : \exists y \in W, x \leq y\}$ is clopen whenever W is clopen

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$$f(x) \leq y \implies \exists! x' \geq x, f(x') = y;$$

equivalently $\uparrow x \cong_f \uparrow f(x)$.

For example:

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The algebra $\mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N}) \oplus 2$ is étale-finite, but is neither complete nor a Boolean algebra – not even a homomorphic image of a product.

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A map $f: (X, \leq, \tau) \rightarrow (Y, \leq, \tau)$ between Priestley spaces is a **spectral local homeomorphism** if f is a local homeomorphism for the spectral topology.

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Proposition (Abbadini, A., Arrieta)

f is a spectral local homeomorphism if and only if it is a local homeomorphism for the Priestley/patch topology, and a strict p -morphism.

Building toposes

Our analogue of $\mathbf{Top}_{\text{LH}}$ – topological spaces with local homeomorphisms – is then

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Not such a nice category: lacks terminal objects and equalizers. But its **slices** are good.

For X an Esakia space:

$$\mathbf{Esa}\acute{\mathbf{E}}\mathbf{t}(X) = \mathbf{Esa}_{\text{SLH}}/X$$

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Theorem (Abbadini, A., Arrieta)

For every Esakia space X , the category $\mathbf{Esa\acute{E}t}(X)$ is finitely complete:

1. (Terminal object) *The identity $1_X: X \rightarrow X$ is the terminal object in $\mathbf{Esa\acute{E}t}(X)$.*

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3. (Equaliser) Whenever $f: Y \rightarrow X$ and $g: Z \rightarrow X$ are two objects of $\mathbf{Esa\acute{E}t}(X)$, and $p_1, p_2: Y \rightarrow Z$ are two spectral local homeomorphisms making the two triangles in the diagram commute,

$$\begin{array}{ccc} Y & \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} & Z \\ & \begin{array}{c} \searrow f \\ \swarrow g \end{array} & \\ & X & \end{array}$$

then

$$\text{Eq}(p_1, p_2) = \{y \in Y : p_1(y) = p_2(y)\},$$

together with the inclusion into Y , is the equaliser of p_1 and p_2 in $\mathbf{Esa\acute{E}t}(X)$.

Any inclusion of a **clopen upset** $U \hookrightarrow X$ is a spectral local homeomorphism, and these are all the injective ones. So we have:

Theorem (Abbadini, A., Arrieta)

Given an Esakia space, $\mathcal{E} = \mathbf{Esa}\acute{\mathbf{E}}\mathbf{t}(X)$, we have:

$$\mathbf{Sub}_{\mathcal{E}}(1) \cong \mathbf{ClopUp}(X).$$

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Via Esakia duality, this gives us **every Heyting algebra**. Incidentally, $\mathbf{Esa\acute{E}t}(X)$ is always a **Heyting pretopos**.

What about power objects?

What about **power objects**?

More difficult to construct. Basic idea: use a **Vietoris-like** construction (typical in the theory of (co)free objects in theories such as Heyting algebras and modal algebras).

Given $f: Y \rightarrow X$ an object in $\mathbf{Esa\acute{E}t}(X)$, where X is étale-finite:

$$P_f(Y) := \{(x, K) : x \in X, K \text{ an upset of } Y \text{ with } K \subseteq f^{-1}[\uparrow x]\}.$$

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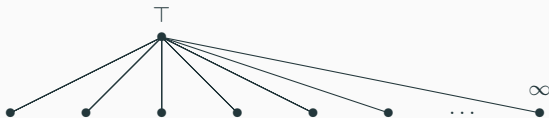
we define the partial order $\preceq$ by setting

$$(x, K) \preceq (x', K') \iff x \leq x' \text{ and } K' = K \cap f^{-1}[\uparrow x'].$$

and let it inherit the topology from the product $X \times \mathcal{V}(Y)$ – the Vietoris space of Y .

Power objects – Example

Take the space:

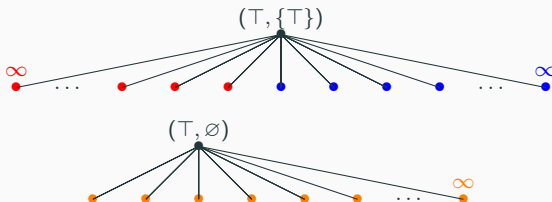


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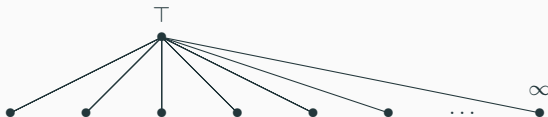


Then $P_{1_X}(X)$ will be:

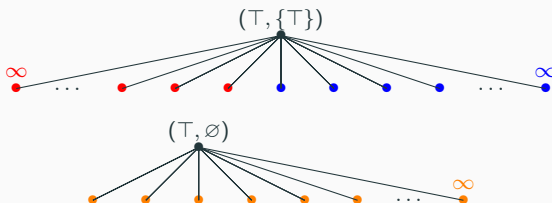


Power objects – Example

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Blue points – $(x, \{T\})$;

Red points – $(x, \{x, T\})$.

Orange points – $(x, \emptyset)$.

Theorem (Abbadini, A., Arrieta)

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The key difficulty lies in showing that $P_f(Y)$ is actually a **compact space**. But with that we obtain:

Theorem (Elementary topos for étale-finite)

For every étale-finite Heyting algebra H , with $X = \text{Spec}(H)$, $\mathcal{E} = \mathbf{Esa\acute{E}t}(X)$ is an elementary topos, such that $\text{Sub}_{\mathcal{E}}(1) \cong H$.

Finitely propositional toposes

Two interrelated questions:

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Definition ((Finitely) propositional)

A category (with a terminal object) is **(finitely) propositional** if every object Y admits a jointly epimorphic (finite) family of morphisms from subterminal objects

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Theorem (Abbadini, A., Arrieta)

For every Esakia space X , the category $\mathbf{EsaEt}(X)$ is finitely propositional.

Through studying the structure of the lattice of truth-values, we get (Abbadini, A., Almeida):

Theorem (Finitely propositional $\leftrightarrow$ étale-finite)

For a Heyting algebra H , the following are equivalent:

1. $\mathbf{Esa}\mathbf{\acute{E}t}(\mathrm{Spec}(H))$ is a topos;
2. H is isomorphic to the lattice of truth values of some finitely propositional topos;
3. H is étale-finite.

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This hints at a possible (2-)equivalence of categories, which we have not yet investigated.

Final remarks

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2. It would be interesting to approach a characterization more tightly: identify stronger necessary conditions.
3. Interaction with other topos-theoretic structures: what happens if we ask that the topos has an NNO? Or that the topos is scattered?

Thank you!



Questions?