

CATEGORIES WITH INDEXED MONOIDS

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Uniform & natural (co)monoids

Definition 1. A **uniform monoid** in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is a choice of a monoid

$$(m_x : x \otimes x \rightarrow x, u_x : I \rightarrow x)$$

for each $x \in \mathcal{Ob}(\mathcal{C})$ such that $m_I = \rho_I$, $u_I = \text{id}_I$ and

$$\begin{array}{c} x \\ y \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} m_{x \otimes y} \\ \text{---} \end{array} \begin{array}{c} x \\ y \end{array} = \begin{array}{c} x \\ y \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} m_x \\ \text{---} \end{array} \begin{array}{c} x \\ y \end{array} \quad \begin{array}{c} x \\ y \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} u_{x \otimes y} \\ \text{---} \end{array} \begin{array}{c} x \\ y \end{array} = \begin{array}{c} x \\ y \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} u_x \\ \text{---} \end{array} \begin{array}{c} x \\ y \end{array} \quad (1) \quad (2)$$

Definition 2. A **natural monoid** is a uniform monoid such that for all morphisms $f : x \rightarrow y$ one has

$$\begin{array}{c} x \\ x \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} m_x \\ \text{---} \end{array} \begin{array}{c} f \\ \text{---} \end{array} \begin{array}{c} y \\ y \end{array} = \begin{array}{c} x \\ x \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} f \\ \text{---} \end{array} \begin{array}{c} m_y \\ \text{---} \end{array} \begin{array}{c} y \\ y \end{array} \quad \begin{array}{c} x \\ x \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} u_x \\ \text{---} \end{array} \begin{array}{c} f \\ \text{---} \end{array} \begin{array}{c} y \\ y \end{array} = \begin{array}{c} x \\ x \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} u_y \\ \text{---} \end{array} \begin{array}{c} y \\ y \end{array} \quad (3) \quad (4)$$

Uniform and natural comonoids are defined by duality.

Proposition 1. (Fox's theorem) A symmetric monoidal category is cartesian monoidal if and only if it has natural comonoids.

1. Indexed monoids

Definition 3. An **im-category** $(\mathcal{C}, m)$ is a cartesian monoidal category $\mathcal{C}$ with a uniform monoid m with natural units (i.e. equation (4) holds).

Definition 4. A morphism $f : x \rightarrow y$ in an im-category $(\mathcal{C}, m)$ is a **monoid homomorphism** if $m_x; f = (f \otimes f); m_y$ (i.e. equation (3) holds).

The category **IMon** has im-categories as objects; the morphisms $(\mathcal{C}, m) \rightarrow (\mathcal{D}, m)$ are strong monoidal functors $(F, \mu) : \mathcal{C} \rightarrow \mathcal{D}$ such that $\mu_{xx}; F(m_x) = m_{F_x}$ for all $x \in \mathcal{Ob}(\mathcal{C})$.

Proposition 2. Taking the set of monoid homomorphisms extends to a functor $\text{hom} : \mathbf{IMon} \rightarrow \mathbf{Cat}$, which moreover has a left adjoint given by the **free im-category functor** $F_{\text{im}} : \mathbf{Cat} \rightarrow \mathbf{IMon}$.

3. Main theorem

Theorem 3. Let $\mathcal{X}$ be a small category. The category of monoids on split opfibrations with base $\mathcal{X}$ is equivalent to the category of split im-opfibrations with base $F_{\text{im}}(\mathcal{X})$:

$$\mathbf{Mon}_{\text{st}}[\mathbf{OpFib}_{\text{sp}}(\mathcal{X})] \simeq \mathbf{imOpFib}_{\text{sp}}(F_{\text{im}}(\mathcal{X})).$$

Corollary 4.

$$\mathbf{imOpFib}_{\text{sp}}(F_{\text{im}}(\mathcal{X})) \simeq [\mathcal{X}, \mathbf{MonCat}_{\text{st}}].$$

Monoids of opfibrations

The category of opfibrations $\mathbf{OpFib}(\mathcal{X})$ into $\mathcal{X}$ has cartesian monoidal structure given by taking pullbacks:

$$\begin{array}{ccc} \mathcal{Y} \boxtimes \mathcal{Z} & \longrightarrow & \mathcal{Z} \\ \downarrow & \lrcorner & \downarrow q \\ \mathcal{Y} & \xrightarrow{p} & \mathcal{X} \end{array} \quad (y, z) \in \mathcal{Y} \boxtimes \mathcal{Z} \text{ if and only if } py = qz,$$

while the terminal object is given by $\text{id}_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X}$. A monoid in $(\mathbf{OpFib}(\mathcal{X}), \boxtimes, \text{id}_{\mathcal{X}})$ is given by functors

$$\begin{array}{ccc} \mathcal{Y} \boxtimes \mathcal{Y} & \xrightarrow{\otimes} & \mathcal{Y} \\ & \searrow & \swarrow \\ & \mathcal{X} & \end{array} \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{1} & \mathcal{Y} \\ \text{id}_{\mathcal{X}} \searrow & & \swarrow \\ & \mathcal{X} & \end{array}$$

associative and unital up to an isomorphism.

- $\mathbf{Mon}[\mathbf{OpFib}(\mathcal{X})]$ is the category of monoids,
- $\mathbf{Mon}_{\text{st}}[\mathbf{OpFib}_{\text{sp}}(\mathcal{X})]$ is the category of strictly associative and unital monoids on split opfibrations.

Theorem 5. (Moeller and Vasilakopoulou) (special case of a special case)

$$\mathbf{Mon}_{\text{st}}[\mathbf{OpFib}_{\text{sp}}(\mathcal{X})] \simeq [\mathcal{X}, \mathbf{MonCat}_{\text{st}}].$$

2. Opfibrations and indexed monoids

Definition 5. An **im-opfibration** $\mathcal{Y} \rightarrow \mathcal{X}$ is an opfibration such that

- the base category $\mathcal{X}$ is an im-category,
- the total category $\mathcal{Y}$ is cartesian monoidal,
- the functor $\mathcal{Y} \rightarrow \mathcal{X}$ preserves and reflects the cartesian products,
- the cartesian product of opcartesian maps in $\mathcal{Y}$ is opcartesian.

An im-opfibration is **split** if it is a split opfibration and the cartesian products are preserved and reflected strictly.

4. Future work

- What about the non-strict cases? Note that both monoids and opfibrations can be non-strict.
- What do the im-opfibrations with non-free base category correspond to?

