

Bialgebraic Structures from the 2-Categorical Perspective

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1. Motivations

Can general bialgebraic structures be captured using a single unified syntactic framework (Hopf algebras, monad algebras, algebras for monad-comonads)?

The answer is yes.

Each of these structures possesses a 2-categorical universal property constructed strictly from **PIE limits** (Strict Products, Inserters, Equifiers).

This syntactic unification yields two main results: **accessibility** and **local presentability** lift automatically, and 1-categorical calculations reduce to simple 2-dimensional limits. These results are formalized in [3]. Our framework is closely related to Ulmer's notion of bialgebras [5].

2. Syntax: Signature pair (Σ, σ) and σ -theory T

We outline a signature pair (Σ, σ) which can be modeled in 2-category $\mathcal{K}$ with PIE limits.

The 2-Signature Σ (Categorical Background)

Syntax permitting a construction of a free 2-category:

- **Category symbols:** $C = \{c, d, \dots\}$.
- **Functor symbols:** $B: c_1 \dots c_n \rightarrow d \in \mathcal{F}$. For a set V of variable symbols, consider the typed set of $\text{Term}_V^\Sigma \rightarrow C$ of terms.
- **Transformation symbols:** $\eta^{\bar{x}}: t_1 \rightarrow t_2 \in \mathcal{T}$ for $t_1, t_2 \in \text{Term}_V^\Sigma$ of the same type with the variables in context $\bar{x} \in V^*$.

A Σ -model M in 2-category $\mathcal{K}$ with finite products instantiates this universe.

The 1-Signature σ (Internal Algebra)

Defined relative to Σ , specifying the internal algebraic logic:

- **Sorts:** $S \rightarrow C$.
- **Operations:** Function symbols $f: t_1 \rightarrow t_2: c_f$, with the **arity** t_1 and the **coarity** t_2 terms being of the same type c_f .

The σ -Theory T

A σ -theory is a set of pairs $(p, q: t_1 \rightarrow t_2: c)$ of parallel paths in the **term graph** G_σ , where the arities t_1 and coarities t_2 are of type c .

3. Example: The Theory of Monoids

Consider the signature pair $(\Sigma_{\text{Mon}}, \sigma_{\text{Mon}})$ for a monoid within a monoidal structure:

$$\Sigma_{\text{Mon}} = (C = \{c\}, \mathcal{F} = \{\otimes: cc \rightarrow c, I: () \rightarrow c\}, \mathcal{T} = \{\alpha^{xyz}, \lambda^x, \rho^x\})$$
$$\sigma_{\text{Mon}} = (S = \{m: c\}, F = \{\mu: m \otimes m \rightarrow m, u: I \rightarrow m\})$$

Theory of monoids T_{Mon}

$$\begin{array}{ccc} (m \otimes m) \otimes m \xrightarrow{\alpha_{m,m,m}} m \otimes (m \otimes m) \xrightarrow{m \otimes \mu} m \otimes m & I \otimes m \xrightarrow{u \otimes m} m \otimes m \xrightarrow{m \otimes u} m \otimes I & \\ \mu \otimes m \downarrow & \lambda_m \searrow \quad \mu \downarrow \quad \rho_m \swarrow & \\ m \otimes m & \xrightarrow{\mu} & m \end{array}$$

References

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4. Object of bialgebras M^T via strict PIE limits

Consider a signature pair (Σ, σ) and a σ -bialgebraic theory T . The object M^T of T -bialgebras is constructed via strict products, inserters, and equifiers in a 2-category $\mathcal{K}$. We set $M_0 := \Pi_s M_{c_s}$, where indexing is over σ -sorts s .

Step 1: The Object of σ -models (Inserter)

The algebraic structure arises through the inserter M^σ in $\mathcal{K}$:

$$\begin{array}{ccc} M^\sigma & \xrightarrow{U_M} & M_0 \\ U_M \downarrow & \phi_M \nearrow & \downarrow (M(t_2))_f \\ M_0 & \xrightarrow{(M(t_1))_f} & \Pi_f M_{c_f} \end{array}$$

Step 2: The Object of T -models (Equifier)

Each path $p: t_1 \rightarrow t_2: c$ in term graph G_σ produces a 2-cell $\bar{p}: M(t_1)U_M \Rightarrow M(t_2)U_M$ and the equations $(p, q: t_1 \rightarrow t_2: c) \in T$ jointly produce M^T via the equifier:

$$M^T \xrightarrow{e_M} M^\sigma \begin{array}{c} \xrightarrow{(M(t_1)U)_{(p,q)}} \\ \Downarrow (p) \quad \Downarrow (q) \\ \xrightarrow{(M(t_2)U)_{(p,q)}} \end{array} \Pi_{(p,q)} M_c$$

5. Theorems: Limits, Colimits, and Accessibility

Let (Σ, σ) be a signature pair with a σ -theory T . Let $\mathcal{K}$ be a strict 2-category with PIE-limits. Let $\mathcal{V}$ be a locally presentable symmetric monoidal closed category.

Theorem: Strict Creation of Kan Extensions [3]

Let M be a Σ -model in $\mathcal{K}$. The forgetful 1-cell $U: M^T \rightarrow M_0$ strictly creates those

- left Kan extensions (colimits) preserved by $M(t_1)$ for all arity terms t_1 .
- right Kan extensions (limits) preserved by $M(t_2)$ for every coarity term t_2 .

Theorem: Local presentability of Bialgebras in $\mathcal{V}$ -CAT [1, 3, 4, 2]

Let M be a Σ -model in $\mathcal{V}$ -CAT. If each $\mathcal{V}$ -category M_c is accessible and each $\mathcal{V}$ -functor $M(B)$ is accessible, then the forgetful $\mathcal{V}$ -functor $U: M^T \rightarrow M_0$ is accessible. Moreover, if T is (co)algebraic and each M_c is locally presentable, then U is (co)monadic and M^T locally presentable.

Theorem: Monadicity of $M^T \rightarrow M_0$

Let κ be a cardinal and assume (Σ, σ) has at most κ symbols. Assume T is an algebraic theory. Let M be a Σ -model in **CAT**, where each M_c has colimits of diagrams of size at most κ and each functor $M(B)$ preserves colimits of κ -chains componentwise. Then $M^T \rightarrow M_0$ is monadic. Moreover, if M_0 has all left Kan extensions along a functor $k: I \rightarrow K$, then so does M^T .

6. Structural Lifting: Functors of Algebras, OFS, Regularity, and Exactness

Theorem: Induced Functor of Algebras

If $F: M \rightarrow N$ is a strong/lax/oplax morphism of Σ -models in $\mathcal{K}$ and T is a bialgebraic/algebraic/coalgebraic theory, then there is an induced morphism $F^T: M^T \rightarrow N^T$.

Theorem: Lifting of OFS, Regularity and Exactness [3]

1. If each category M_c is equipped with an OFS $(\mathcal{E}_c, \mathcal{M}_c)$ such that $M(t): M_0 \rightarrow M_c$ preserves the left class for each arity term $t: c$ and the right class for each coarity term t , then $(U^{-1}(\mathcal{E}_0), U^{-1}(\mathcal{M}_0))$ is the unique OFS on M^T preserved by the forgetful functor $U: M^T \rightarrow M_0$.
2. Assume each M_c is a regular category and $M(t)$ preserves regular epis componentwise for each arity term t and finite limits for each coarity term t . Then M^T is regular. Furthermore, if each category M_c is exact and each functor $M(B)$ preserves reflexive coequalizers componentwise, then M^T is exact.