

The second Čech homology is simplicial

PSSL113 Louvain-la-Neuve

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1 Main theorem [2]

Consider

$$\mathcal{D} \xleftarrow[\perp]{I} \mathcal{C} \xleftarrow[\perp]{F} \mathcal{X}$$

where $\mathcal{D}$ a Birkhoff subcategory of a semi-abelian category $\mathcal{C}$ and consider the comonad $\mathbb{G}$ derived from $F \dashv U$ such that

- $\varepsilon_C: G(C) \rightarrow C$ is a regular epimorphism for all $C \in \mathcal{C}$;
- for all $C \in \mathcal{C}$, we have that $G(C)$ is a projective object in $\mathcal{C}$ (with respect to regular epimorphisms).

Example : $\mathcal{C}$ is a semi-abelian variety and $\mathcal{D}$ a subvariety of $\mathcal{C}$, $\mathbb{G}$ is the underlying set/free object comonad.

Proposition

From $F \dashv U$ satisfying (1)-(2), for $n \geq 0$, we can define a comonad $\mathbb{G}_n$ on $\text{Arr}^n(\mathcal{C})$ derived from

$$\text{Arr}^n(\mathcal{C}) \xleftarrow[\perp]{F_n} \text{Arr}^n(\mathcal{X})$$

where the functors and the natural transformations are defined level-wise.
Furthermore, $G_n := F_n U_n$ preserves n -extension [6, Proposition 5.6.].

Theorem

$$H_2(C, I)_{\mathbb{G}} \cong \frac{\text{Ker}(I(\partial_0^1)) \cap \text{Ker}(I(\partial_1^1))}{I(\partial_0^2)(\text{Ker}(I(\partial_1^2)) \cap \text{Ker}(I(\partial_2^2)))} \cong L_1(I)(f, C).$$

2 Barr-Beck's cotriple homology [1, 4]

Given a (semi)-simplicial object $\mathbb{C}$, we define

$$\begin{array}{ccccccc} \mathbb{C} \in \mathfrak{s}(\mathcal{C}) & \cdots & C_2 & \xrightleftharpoons[\partial_2^2]{\partial_1^2} C_1 & \xrightarrow{\partial_0^1} C_0 & & \\ & & \downarrow & \downarrow & \downarrow & & \\ N(\mathbb{C}) \in \text{Ch}(\mathcal{C})^+ & \cdots & \text{Ker}(\partial_1^2) \cap \text{Ker}(\partial_2^2) & \xrightarrow{d_2^2} \text{Ker}(\partial_1^1) & \xrightarrow{d_1^1} C_0 & \xrightarrow{0} 0 & \cdots \end{array}$$

Given a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on a category $\mathcal{C}$, we can define a (semi)-simplicial object

$$\mathbb{G}(C) : G^{n+1}(C) \xrightleftharpoons[\sigma_i := G^i(\delta_{C^{n-1-i}(C)})]{\partial_i := G^i(\varepsilon_{G^n \rightarrow i(C)})} G^n(C) \quad \cdots \quad G^2(C) \xleftarrow[\varepsilon_{G(C)}]{\delta_C} G(C) \xrightarrow{\varepsilon_C} C.$$

and for $n \geq 0$, we can define

$$H_{n+1}(C, I)_{\mathbb{G}} := H_n(N(I(\mathbb{G}(C)))) \left(\stackrel{\text{(in short)}}{=} H_n(I(\mathbb{G}(C))) \right).$$

3 Čech resolution [5, 7, 9]

Definition 1. A Čech resolution of C , denoted $\check{C}(f, C)$, is the augmented complex

$$\cdots \xrightarrow{\quad} P \times_C P \times_C P \xrightleftharpoons[\partial_2^2]{\partial_1^2} P \times_C P \xrightarrow[\partial_1^1]{\partial_0^1} P \xrightarrow{f} C$$

where $f: P \rightarrow C$ is a projective presentation of C . Hence $\check{C}(f, C)_0 = P$, $\check{C}(f, C)_1 = P \times_C P$ and, for $n \geq 2$,

$$\check{C}(f, C)_n := \underbrace{P \times_C \cdots \times_C P}_{(n+1) \text{ factors}}$$

is the n -fold pullback.

For us the important part is

$$\begin{array}{ccccc} P \times_C P \times_C P & \xrightarrow{\partial_2^2} & P \times_C P & \xrightarrow{\partial_0^1} & P \\ \downarrow \partial_1^2 & \searrow \partial_0^2 & \downarrow \partial_1^1 & \searrow \partial_0^1 & \downarrow f \\ P \times_C P & \xrightarrow{\partial_1^2} & P & \xrightarrow{\partial_0^1} & P \\ \downarrow \partial_0^2 & \searrow \partial_1^1 & \downarrow \partial_0^1 & \searrow \partial_1^0 & \downarrow f \\ P & \xrightarrow{\partial_1^1} & P & \xrightarrow{\partial_0^1} & C \end{array} \quad (1)$$

where all the faces denote a pullback. In particular, we have that

- $(\partial_0^1, \partial_1^1)$ is the kernel pair of f ;
- $(\partial_0^2, \partial_1^2)$ is the kernel pair of ∂_0^1 ;
- $(\partial_1^2, \partial_2^2)$ is the kernel pair of ∂_1^1 .

Remark

By construction, the cube (1) determines a 3-fold extension. Applying $\mathbb{G}$, we obtain a cube of simplicial resolutions which are level-wise 3-fold extension in $\mathcal{C}$.

Example : (Category of sets)

For $n \geq 0$

$$\check{C}(f, C)_n := \underbrace{P \times_C \cdots \times_C P}_{(n+1)\text{-factors}} = \{(x_0, \dots, x_n) \in P^{n+1} \mid f(x_0) = \cdots = f(x_n)\}$$

and for face operators $\partial_i^n: \check{C}(f, C)_n \rightarrow \check{C}(f, C)_{n-1}$ and for the degeneracy operators $\sigma_i^n: \check{C}(f, C)_n \rightarrow \check{C}(f, C)_{n+1}$: for $0 \leq i \leq n$

$$\begin{aligned} \partial_i^n(x_0, \dots, x_n) &= (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ \sigma_i^n(x_0, \dots, x_n) &= (x_0, \dots, x_i, x_i, x_{i+1}, \dots, x_n). \end{aligned}$$

Proposition

$$L_1(I)(f, C) := H_1(N(I(\check{C}(f, C)))) \cong \frac{\text{Ker}(I(\partial_0^1)) \cap \text{Ker}(I(\partial_1^1))}{I(\partial_0^2)(\text{Ker}(I(\partial_1^2)) \cap \text{Ker}(I(\partial_2^2)))}$$

Proof. Follow from the definition of $\check{C}(f, C)$ and of the functor N . $\square$

4 Barr-Beck simplicial approach to H_2

Proposition

$$H_2(C, I)_{\mathbb{G}} = H_1(I(\mathbb{G}(C))) \cong \frac{\text{Ker}(I(\partial_0^1)) \cap \text{Ker}(I(\partial_1^1))}{I(\partial_0^2)(\text{Ker}(I(\partial_1^2)) \cap \text{Ker}(I(\partial_2^2)))}$$

From (1), we obtain:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(I(\mathbb{G}(\partial_2^2))) & \longrightarrow & I(\mathbb{G}(P \times_C P \times_C P)) & \xrightarrow{I(\mathbb{G}(\partial_2^2))} & I(\mathbb{G}(P \times_C P)) \longrightarrow 0 \\ & & \downarrow I(\mathbb{G}(\partial_0^2)) & & \downarrow I(\mathbb{G}(\partial_1^2)) & & \downarrow I(\mathbb{G}(\partial_1^2)) \\ 0 & \longrightarrow & \text{Ker}(I(\mathbb{G}(\partial_1^1))) & \longrightarrow & I(\mathbb{G}(P \times_C P)) & \xrightarrow{I(\mathbb{G}(\partial_1^1))} & I(\mathbb{G}(P)) \longrightarrow 0 \\ & & \downarrow I(\mathbb{G}(\partial_0^1)) & & \downarrow I(\mathbb{G}(f)) & & \downarrow I(\mathbb{G}(f)) \\ 0 & \longrightarrow & \text{Ker}(I(\mathbb{G}(f))) & \longrightarrow & I(\mathbb{G}(P)) & \xrightarrow{I(\mathbb{G}(f))} & I(\mathbb{G}(C)) \longrightarrow 0 \end{array} \quad (2)$$

Using results from [6], we can deduce most of the technical results from the above diagram and in particular, we have

$$\begin{array}{ccccc} I(\partial_0^2)(\text{Ker}(I(\partial_1^2) \cap \text{Ker}(I(\partial_2^2)))) & \longrightarrow & I(\partial_0^2)(\text{Ker}(I(\partial_1^2) \cap \text{Ker}(I(\partial_2^2)))) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \\ \text{Ker}(I(\partial_0^1)) \cap \text{Ker}(I(\partial_1^1)) \cong \text{Ker}(I(\partial_0^1)) & \longrightarrow & \text{Ker}(I(\partial_1^1)) & \xrightarrow{I(\partial_0^1)} & \text{Ker}(I(f)) \\ & & \downarrow & & \downarrow \\ H_1(I(\mathbb{G}(C))) & \xrightarrow{\delta} & H_0(\text{Ker}(I(\mathbb{G}(f)))) & \xrightarrow{q} & \text{Ker}(I(f)) \end{array}$$

where

- q is the unique morphism such that $H_0(\text{ker}(I(\mathbb{G}(f)))) = \text{ker}(I(f))q$;
- the bottom horizontal sequence is a short exact sequence (long exact sequence associated to the bottom row of (2)) where δ is “connecting” morphism;
- since $I(\mathbb{G}(\partial_0^1))$ is the coequalizer of $(I(\mathbb{G}(\partial_0^2)), I(\mathbb{G}(\partial_1^2)))$ (we use the **Remark** for the proof), we can deduce that the middle vertical sequence is a short exact sequence.

By (3×3) -Lemma, the left-hand vertical sequence is a short exact sequence and the claim follows by Noether's First Isomorphism Theorem:

$$H_2(C, I)_{\mathbb{G}} = H_1(I(\mathbb{G}(C))) \cong \frac{\text{Ker}(I(\partial_0^1)) \cap \text{Ker}(I(\partial_1^1))}{I(\partial_0^2)(\text{Ker}(I(\partial_1^2)) \cap \text{Ker}(I(\partial_2^2)))}.$$

5 Main consequence of the theorem

The first Čech derived functor of I is well-defined: it is independent of the choice of projective presentation.

References

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