Properties of the algebraic theories all of whose algebras are free

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Each algebraic theory (equivalently finitary monad on **Set**) has some free algebras. Which algebraic theories have the property that all their algebras are free? It turns out that it is not hard to state precisely which theories have this property: they are the theories of sets with at most one element, of sets with exactly one element (these first two I will call *degenerate* theories), of sets, of pointed sets, of affine spaces over a given division ring, and of vector spaces over a given division ring.

Proving that these are the only such theories is a much more difficult task. Proofs were given in the 1970s in the language of logic by Steven Givant [1], and in the 2010s in the language of classical universal algebra by Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei [2]. Tom Leinster and I investigated this result from a category theoretic point of view [3].

I will talk about the theories in question, highlighting some of their interesting properties: the theories are split into affine (exactly one unary operation) and pointed (exactly one nullary operation) theories, each nondegenerate theory has a notion of dimension (its free functor is essentially injective on objects), and each nondegenerate theory is minimal (i.e. simple: quotienting it/adding in more equations either does nothing, or collapses it down to a degenerate theory). We found category theoretic proofs of these latter two properties that do not depend on the classification theorem.

References

[1] S. Givant, Universal Horn classes categorical or free in power, *Annals of Mathematical Logic* **15** (1978), 1–53.

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[3] M. Woolf, *Algebraic theories all of whose algebras are free*, MScR thesis, University of Edinburgh, 2025.