

From group gradings to 2-group gradings: categories and invariants of flat 2-bundles

Kürşat Sözer^{1,*} and Alexis Virelizier¹

¹Université de Lille, CNRS, UMR 8524, Laboratoire Paul Painlevé, F-59000 Lille, France

*Speaker

Group-graded monoidal categories arise naturally in constructions of invariants of manifolds equipped with principal bundles. We will describe a categorification of this framework in which a group is replaced by a crossed module $\chi: E \longrightarrow H$, or equivalently by a strict 2-group. The classifying space $B\chi$ is a connected homotopy 2-type, and homotopy classes of maps $M \rightarrow B\chi$ classify flat principal 2-bundles over M .

The corresponding algebraic structure is a monoidal category graded by χ . Here objects have degrees in H , while morphisms have degrees in E , and the two gradings are related through χ . Spherical fusion categories with such a grading produce 3-dimensional state-sum homotopy quantum field theories with target $B\chi$ [2]. We will also explain how Hopf algebras graded by crossed modules provide concrete examples through their representation categories [3].

The main application is an invariant of flat principal 2-bundles over closed oriented 3-manifolds [4]. Maps $M \rightarrow B\chi$ are represented combinatorially by crossed-module colored Heegaard diagrams, and a finite-type involutory Hopf χ -coalgebra assigns a scalar to such a diagram. The resulting invariant extends Kuperberg's invariant of 3-manifolds [1] and Virelizier's refinement for flat principal G -bundles [5].

References

- [1] G. Kuperberg, Involutory Hopf algebras and 3-manifold invariants, *Internat. J. Math.* **2** (1991), 41–66.
- [2] K. Sözer and A. Virelizier, Monoidal categories graded by crossed modules and 3-dimensional HQFTs, *Adv. Math.* **428** (2023), 109155.
- [3] K. Sözer and A. Virelizier, Hopf crossed module (co)algebras, *J. Pure Appl. Algebra* **229** (2025), no. 11, 108083.
- [4] K. Sözer and A. Virelizier, Quantum invariants of flat 2-bundles over 3-manifolds, arXiv:2605.22405, 2026.
- [5] A. Virelizier, Involutory Hopf group-coalgebras and flat bundles over 3-manifolds, *Fund. Math.* **188** (2005), 241–270.