A Universal Property for Measure-theoretic Probability

Eigil Fjeldgren Rischel[1]

[1]Tallinn University of Technology

There is a large and growing body of work applying the methods of category theory to probability theoretic questions. One prominent version of this is the *synthetic* approach, which attempts to describe probability theory in abstract terms, and deduce results from this description. Here the focus is usually on characterizing the monad P which assigns to each space X (set, type, or whatever domain is under consideration) the space PX of probability distributions on X—or, what is largely equivalent, of characterizing the Kleisli category of this monad. One example of this approach is the school of Markov categories [2, 3, 4], which tries to axiomatize the structure of an abstract monoidal category which allows one to state and prove theorems of probability theory.

In my talk, I will present some recent work giving a strong comparison between the theory of Markov categories and "real" probability theory done with σ-algebras. Specifically, we have the following theorem:

Theorem 1 (paraphrased)**.** *Among Markov categories which are (in a suitable sense) countably extensive, countably complete, and Boolean, and which admit a "fair coinflip" morphism* $1 \to 1 + 1$ *(that is, one which is symmetric), the category* $\mathsf{BorelStoch}$ *of standard Borel measurable spaces and Markov kernels between them is initial.*

This builds on work of Chen [1], who proved that the category of standard Borel spaces (and ordinary measurable functions) is initial among countably extensive, countably complete, Boolean categories, and functors which preserve countable limits and countable coproducts. Thus the meat of the above result is that adding the single nondeterministic morphism $1 \to 1 + 1$ suffices to generate all the Markov kernels in ordinary probability theory, and moreover that every equation between these can be proven using just the general assumptions on the category (as well as the single equation demanding that the coinflip be symmetric). Apart from Chen's theorem, the main ingredient in the proof is the concept of *Kolmogorov product* (from [3]), which gives a suitable universal property for infinite-dimensional products in the context of probability theory.

References

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[4] Tobias Fritz et al. *Empirical Measures and Strong Laws of Large Numbers in Categorical Probability.* Mar. 2025. DOI: `10.48550/arXiv.2503.21576`.