Skew structures as multiactions

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Skew monoidal categories are a generalisation of monoidal categories in which the structural natural transformations

$$\begin{aligned} \alpha_{a,b,c} &: (a \otimes b) \otimes c \to a \otimes (b \otimes c) \\ \lambda_a &: i \otimes a \to b \\ \rho_a &: a \to a \otimes i \end{aligned}$$

are not required to be invertible. Skew monoidal categories were independently discovered in [1] and [8] and, since then, they have been extensively studied as more examples and applications emerged.

Many examples of skew monoidal categories seem to arise from **(skew) actions of (skew) monoidal categories**. A common theme is that when an action is equipped with a suitable additional structure, it induces a skew monoidal structure on the actegory [4, 6, 5].

In this talk, we demonstrate that **multiactions and their morphisms** provide a unifying context in which all the examples of interest can be characterised. A multiaction $\mathbf{A}$ is a structure with maps graded by lists of objects of some multicategory $\mathbf{M}$, i.e. comprising multihomsets

$$\mathbf{A}(a, m_1, \ldots, m_k; a'), \quad a, a' \in \mathbf{A}, k \geq 0, m_1, \ldots, m_k \in |\mathbf{M}|.$$

To capture skew structures, we develop **representability theory for multiactions** and their morphisms. (A systematic study of representable multiactions appears to have been missing in the literature, although special cases of our general result have appeared e.g. in [7].)

Representability theory for multicategories, due to Hermida [3], allows one to capture monoidal categories in the language of multicategories. An analogous result for skew monoidal categories was given in [2], where the notion of skew multicategory was introduced as well.

We refine these results by observing that a skew multicategory of [2] may be equivalently defined as an $\mathbf{M}$-multiaction $\mathbf{A}$ together with an identity on objects $\mathbf{M}$-multiaction morphism $\mathbf{A} \to \mathbf{M}$. By rephrasing the notion of *left representability* appearing *ibid.* and by considering more general morphisms of form $\mathbf{A} \to \mathbf{M}$, we capture all the aforementioned constructions of skew monoidal categories, as well as new examples.

References

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