Higher Day convolution over sketches and patterns

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Day convolution is a very useful tool in (higher) categorical algebra. If $\mathcal{V}$ and $\mathcal{W}$ are two symmetric monoidal (∞-)categories, Day convolution is a certain symmetric monoidal structure on the functor category $\mathsf{Fun}(\mathcal{V}, \mathcal{W})$, provided that $\mathcal{W}$ is cocomplete and the tensor product on $\mathcal{W}$ preserves colimits in each variable. This structure enjoys many favorable properties: for example, symmetric lax monoidal functors $\mathcal{V} \to \mathcal{W}$ are commutative algebras with respect to the Day convolution.

In the first part of the talk, I explain a generalization of the classical story to higher dimensions: if $\mathbb{C}$ is any symmetric monoidal (∞, n)-category and $\mathcal{V}$, $\mathcal{W}$ any two E_∞-monoids in $\mathbb{C}$, then there is a certain Day convolution structure on the mapping $(\infty, n-1)$-category $\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}, \mathcal{W})$, making it into an $(\infty, n-1)$-operad. Notice we do not need any assumption on $\mathcal{W}$; the price for that is that the Day convolution is only an operad, not necessarily a monoidal structure. I outline what assumptions one has to put on $\mathcal{W}$ (and $\mathbb{C}$) to remedy this, as well as some examples of this structure in nature. These results are proved in the preprint [1].

The second part of the talk reports on a follow-up work in progress: we will see that the Day convolution exists in much greater generality, for algebraic structures beyond monoids! In [1], this is proved for operads, but we want to capture more general structures such as double categories – for that, I use sketches and algebraic patterns. The generalized form of Day convolution is a bit more difficult to describe, so I end this abstract with an example of what we obtain for double categories: if $\mathbb{C}$ is a "double $(\infty, 2)$-category" (i.e. a Segal object in $\mathbf{Cat}_{(\infty,2)}$) and $\mathcal{V}$, $\mathcal{W}$ are two horizontal monads in $\mathbb{C}$, then $\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}, \mathcal{W})$ admits a Day convolution structure which makes it into a virtual double ∞-category.

References

[1] Tomáš Perutka, 2-dimensional Lawvere theories, commutativity, and higher Day convolution, arXiv:2602.14332, 2026.