A Characterization of Abelianness via Lawvere Doctrines

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Lawvere doctrines have been widely employed in categorical logic in recent years. Several important categorical notions, such as those of regular or coherent category, as well as that of elementary topos, admit characterizations in terms of the existence of a doctrine satisfying suitable properties. For instance, a category $\mathcal{C}$ is regular if and only if there exists an elementary existential doctrine $P : \mathcal{C}^{op} \to \mathit{InfSL}$ with full strong comprehensions and extensional equality satisfying the Rule of Unique Choice ([4]).

The main goal of this talk is to present analogous characterizations using Lawvere doctrines of abelian and semi-abelian categories. To achieve this result, we build on the recent characterization of these categories in terms of forms, developed by Zurab Janelidze ([2], [3]). As a starting point, we consider the notion of orean doctrine, which captures exactly the minimal structure required to abstract kernels, images, and their dual notions for doctrines. We are then able to prove the following characterizations:

Theorem. Let $\mathcal{C}$ be a category with finite products and coproducts. The following are equivalent:

(i) $\mathcal{C}$ is abelian;

(ii) $\mathcal{C}$ admits a binormal noetherian form $F : \mathcal{B} \to \mathcal{C}$;

(iii) $\mathcal{C}$ admits an orean doctrine $P : \mathcal{C}^{op} \to \mathit{Pos}$ satisfying a compatibility condition (CC), with all comprehensions and co-comprehensions satisfying the Beck-Chevalley condition.

Theorem. Let $\mathcal{C}$ be a pointed category with finite products and coproducts. The following are equivalent:

(i) $\mathcal{C}$ is semi-abelian;

(ii) $\mathcal{C}$ admits a conormal noetherian form;

(iii) $\mathcal{C}$ admits an orean doctrine $P : \mathcal{C}^{op} \to \mathit{Pos}$ satisfying (CC), with all comprehensions and the co-comprehensions of normal elements, and these satisfy the Beck-Chevalley condition.

The compatibility condition (CC) is the request that the factorization systems induced by the doctrine P and by its dual P^{op} coincide; intuitively, it ensures that the logical notions of injectivity and surjectivity induced by P correspond to their algebraic counterparts (such as having trivial kernel).

In the end, we discuss potential applications of exact completion of doctrines in this algebraic setting, e.g. its relation with the semi-abelian exact completion of [1].

This talk is based on the Master's Thesis I am writing under the supervision of Prof. Davide Trotta.

References

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