Naturally Mal'tesv $(\infty, 1)$-Categories

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Naturally Mal'tsev categories, initially codified by Johnstone in [2], are of interest in categorical algebra, being situated between affine categories (e.g. slices of additive categories) and Mal'tsev categories (e.g. the category of groups [4]).

Definition 0.1 *([2] A finitely complete category* $\mathbf{C}$ *is* naturally Mal'tsev *if it is equipped with a natural transformation* $p : Id_{\mathbf{C}}^3 \to Id_{\mathbf{C}}$*, such that for each object* $A \in \mathbf{C}$*, the component of* p*,* $p_A : A^3 \to A$*, is a* Mal'tsev operation *on* A*, i.e.* $p(x, x, y) = y$ *and* $p(x, y, y) = x$*.*

These categories admit many equivalent characterizations, including the following, from [2, 4, 3].

Proposition 0.1 *Let* $\mathbf{C}$ *be a finitely complete category. Then the following are equivalent.*

1. $\mathbf{C}$ *is naturally Mal'tsev.*
2. *The fibers* $Pt_A(\mathbf{C})$*, for every object* $A \in \mathbf{C}$*, of the fibration of points* $p : Pt(\mathbf{C}) \to \mathbf{C}$*, are additive.*
3. *The forgetful functor* $U : Cat(\mathbf{C}) \to Refl(\mathbf{C})$*, from the category of internal categories of* $\mathbf{C}$ *to the category of internal reflexive graphs of* $\mathbf{C}$ *is an equivalence.*
4. *The forgetful functor* $U' : Gpd(\mathbf{C}) \to Refl(\mathbf{C})$ *originating at the category of internal groupoids of* $\mathbf{C}$ *is an equivalence.*

In this presentation, we will introduce an $(\infty, 1)$-categorical (henceforth *infinity categorical*) version of naturally Mal'tsev categories, defined as a direct infinity-categorical generalization of the 1-categorical case. These satisfy an analogous list of equivalent characterizations, including the following.

Proposition 0.2 *Let* $\mathscr{C}$ *be a finitely complete infinity category. The following conditions are then equivalent:*

1. $\mathscr{C}$ *is naturally Mal'tsev.*
2. *The fibers* $Pt_A(\mathscr{C})$*,* $A \in Ob(\mathscr{C})$*, of the fibration of points* $p : Pt(\mathscr{C}) \to \mathscr{C}$ *are additive.*
3. *For* $\mathscr{C}$ *also finitely cocomplete,* $\mathscr{C}^{\Delta^{op}_{<2}} \simeq scosk_n(sk_2(\mathscr{C}))$ *is an equivalence of subcategories of* $\mathscr{C}^{\Delta^{op}}$*.*
4. *For* $\mathscr{C}$ *also finitely cocomplete,* $\mathscr{C}^{\Delta^{op}_{<2}} \simeq gcosk_n(sk_2(\mathscr{C}))$ *is an equivalence of subcategories of* $\mathscr{C}^{\Delta^{op}}$*.*

These categories also satisfy, as their one-categorical counterparts do, the "equation" "pointed + naturally Mal'tsev = additive" [2], for an appropriate notion of additivity, and, for the infinity-categorical version of Barr exactness from [5], the "equation" "Barr exact + pointed + naturally Mal'tsev = pre-stable", giving us some examples, with which we shall conclude.

References

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