Internal algebras in weak closed categories

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It is well known that a split extension of groups

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

is determined up to isomorphism by an action of the group B on the group X, which in turn is determined by a homomorphism from B into $\mathrm{Aut}(X)$, the automorphism group of X.

Based in part on a previous work of G. Janelidze and D. Bourn in [3], F. Borceux G. Janelidze, and G.M. Kelly, showed in the semi-abelian [8], context that there is an equivalence of categories between the category of split extensions and the category of *internal object actions* [1]. They also introduced and studied action representable categories which are semi-abelian categories in which each functor $\mathrm{Act}(B, -)$ which assigns to each object the set of actions of B on X, is representable. Examples of action representable categories include the category of groups (mentioned above), the category of Lie algebras over a commutative ring [1], the category of commutative Von Neumann regular rings, as well as the category of Boolean rings [2]. Other examples include categories of internal groups in toposes [2], categories of internal Lie objects in abelian monoidal closed categories [5] as well as functor categories of action representable categories provided they admit normalizers [6]. If drop the semi-abelian requirement in the definition of action representable category, and require insted pointed finitely complete protomodulaity and replace the functors $\mathrm{Act}(B, -)$ by the equivalent (whenever both exist) $\mathrm{SplExt}(B, -)$ functors, then examples include each category of internal groups in a finitely complete cartesian closed category.

The category of (not necessarily unitary) rings is not action representable but was shown to satisfy a weakening introduced by G. Janelidze in [7] called weak action representable. There are two immediate ways one might try and generalize this result: (i) to categories of internal rings in cartesian closed categories; (ii) to categories of internal semi-groups in additive monoidal closed categories.

Although internal abelian groups in a cartesian closed category do not necessarily admit a monoidal structure, they nevertheless admit "internal homs". We introduce weak closed categories, generalizing S. Eilenberg and G. M. Kelly's closed categories [4], which provide a context which includes the above example as as special case. Other examples arise from monoidal closed categories. This allows us to prove a theorem which simultaniously generalizes (i) and (ii) above.

References

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