The second Čech homology is simplicial

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If we consider $\mathcal{B}$ a Birkhoff subcategory of a semi-abelian category $\mathcal{A}$ with a comonad $\mathbb{G}$ derived from an adjunction which must satisfied some reasonable assumptions (such as the classical context when $\mathcal{A}$ is a semi-abelian variety–in the sense of universal algebra, as in [3]) then the authors of [7] give explicit expression for Barr-Beck derived functor [1, 9] of I (the reflector of $\mathcal{A}$ onto $\mathcal{B}$). It turns that such expression are generalization of the famous Hopf's formulae: if we consider $\mathcal{A} = \mathsf{Gp}$ and $\mathcal{B} = \mathsf{Ab}$ with I the abelianisation functor, then we recover the Hopf's formula for the second homology group [10] and also for higher homology group [2].

In this talk, we will assume the same assumptions and we will give an explicit expression of first Čech derived functor [11, 12] of I (the reflector of $\mathcal{A}$ onto $\mathcal{B}$). With this new expression, we can prove that it is equivalent to the Hopf's formula for the second homology object associated to the reflector.

As a consequence, the first Čech derived functor of I turns out to be—like the Hopf's formulae [7]—Baer invariant [8, 9] (i.e. independent of the chosen projective presentation considered to construct the Čech resolution). Furthermore, with extra assumptions, we can prove that the first Čech derived functor of I is naturally isomorphic to the first non-additive left derived functor of I in the sense of [6] (where [4] provides a concrete example of this result).

The main reference of this talk is [5].

References

[1] M. Barr and J. Beck, Homology and standard constructions, Seminar on triples and categorical homology theory (ETH, Zürich, 1966/67), in *Lecture Notes in Math.*, vol. 80, Springer, (1969), pp. 245–335.

[2] R. Brown and G.J. Ellis, Hopf formulae for the higher homology of a group, *Theory Appl. Categ.* **20** (1998), no. 2, 124–128.

[3] D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras, *Theory Appl. Categ.* **11** (2003), no. 6, 143–147.

[4] M. Culot, Projective crossed modules in semi-abelian categories, *Appl. Categ. Structures*, 34, (2026).

[5] M. Culot and E. Everaert, The second Čech homology is simplicial (in preparation), (2026).

[6] M. Culot, F. Renaud, and T. Van der Linden, *Non-additive derived functors via chain resolutions*, Glasgow Math. J. **67** (2025), no. 3, 423–466.

[7] T. Everaert and M. Gran and T. Van der Linden, Higher Hopf formulae for homology via Galois Theory, *Adv. Math.* **217** (2008), 2231–2267.

[8] T. Everaert and T. Van der Linden, Baer invariants in semi-abelian categories I: General theory, *Theory Appl. Categ.* **12** (2004), no. 1, 1–33.

[9] T. Everaert and T. Van der Linden, Baer invariants in semi-abelian categories II: Homology, *Theory Appl. Categ.* **12** (2004), no. 4, 195–224.

[10] H. Hopf, Fundamentalgruppe und zweite Bettische Gruppe, *Comment. Math. Helv.* **14** (1942), 257–309.

[11] H. N. Inassaridze, Non-abelian homological algebra and its applications, *Kluwer Acad. Publ.*, Dordrecht, (1997).

[12] T. Pirashvili, On non-abelian derived functors, *Proc. A.Razmadze Math. Inst., Georgian Acad. Sci.* **42** (1979), 91–104, in Russian.