A topos for étale-finite Heyting algebras

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In [2], Andrew Pitts proved the uniform interpolation property for propositional intuitionistic logic, out of an (ultimately unsuccessful) attempt to provide a negative answer to the following question:

Question 1 (Heyting-to-topos problem) *Is every Heyting algebra isomorphic to the lattice* $\mathrm{Sub}_{\mathcal{E}}(1)$ *of subobjects of the terminal object* 1 *of some elementary topos* $\mathcal{E}$*?*

The lattice $\mathrm{Sub}_{\mathcal{E}}(1)$ is also known as the *lattice of truth values of* $\mathcal{E}$. Two classes of Heyting algebras have been shown to be *topos-admissible*, i.e., isomorphic to $\mathrm{Sub}_{\mathcal{E}}(1)$ for some elementary topos $\mathcal{E}$: (1) *Complete Heyting algebras* (i.e., locales), where one can take the topos of sheaves on the given locale; (2) *Boolean algebras*, for which a construction of Peter Freyd is known to produce a topos with the desired properties (see [1, Exercise 9.11]). Freyd's approach is strongly geometric and heavily reminiscent of the methods of Grothendieck topos theory. Given a Boolean algebra B, Freyd's elementary topos is constructed as follows: letting X denote the Stone space dual to B, one takes the topos of local homeomorphisms $Y \to X$ with Y a Stone space.

In this paper[1], we identify a class of Heyting algebras for which a generalisation of Freyd's approach works – so-called "étale-finite Heyting algebras":

Definition 1 (Étale-finite Heyting algebra) *A Heyting algebra* H *is étale-finite if there is a finite Heyting algebra* H_0 *and a Heyting algebra homomorphism* $f\colon H_0 \to H$ *such that for every* $x \in H$*:*

$$\bigvee_{x_0 \in H_0} (x \leftrightarrow f(x_0)) = 1.$$

Our main tool is the duality theory of Heyting algebras. This is given by considering the category **Esa** of *Esakia spaces*, ordered compact topological spaces, with the duality being witnessed through a *spectrum functor* $\mathsf{Spec}\colon \mathbf{HA}^{\mathrm{op}} \to \mathbf{Esa}$. From this category we extract a (non-full) subcategory $\mathbf{Esa}_{\mathrm{SLH}}$, where maps are *spectral local homeomorphisms*: continuous order-preserving maps that are local homeomorphisms with respect to the topology and the order. Similar to the construction of sheaves using étale spaces, we work with the slice category $\mathbf{Esa}_{\mathrm{SLH}}/X$, where X is an Esakia space, the slice category.

Having outlined the key technology, our main result is the following:

Theorem 1 *Let* H *be a Heyting algebra, and* $X = \mathsf{Spec}(H)$ *be the dual Esakia space:*

1. $\mathbf{Esa}_{\mathrm{SLH}}/X$ *is a Heyting pretopos such that* $\mathrm{Sub}_{\mathcal{E}}(1) \cong H$*.*
2. $\mathbf{Esa}_{\mathrm{SLH}}/X$ *is an elementary topos if and only if* H *is étale-finite.*

In particular, such a result extends the class of Heyting algebras known to be topos-admissible.

References

[1] P. T. Johnstone, Topos theory, Academic Press, 1977.

[2] A. M. Pitts, On an interpretation of second order quantification in first order intuitionistic propositional logic, *Journal of Symbolic Logic* **57** (1992), 33–52.

[3] M. Abbadini, R.N. Almeida, I. Arrieta, A topos for étale-finite Heyting algebras, arxiv:2606.03861, 2026.

[1]The results reported here are collected at [3].