Graduated Categories of Presheaves

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We study properties of presheaves on a small category $\mathcal{A}$. Finitely presentable presheaves are precisely the finite colimits of hom-functors. When do they coincide with the finitely generated ones? For a poset $\mathcal{A}$ this holds iff every ascending chain and every antichain with an upper bound is finite. In case $\mathcal{A}$ is a group, this is true iff $\mathcal{A}$ is Noetherian: every ascending chain of subgroups is finite. Example: the group $\mathbb{Z}$ of integers.

A locally finitely presentable category $\mathcal{K}$ is *graduated* if every finitely presentable object K has all subobjects and all strong quotients finitely presentable, and a grade in $\mathbb{N}$ is assigned to K, greater than the grade of any proper subobject or strong quotient. This concept, introduced in [1], has various consequences for finitary endofunctors F on $\mathcal{K}$: it simplifies the construction of the terminal coalgebra ([2]), and implies that F is a right adjoint iff it preserves countable limits ([1]).

We prove that for a group $\mathcal{A}$ the presheaf category is graduated iff $\mathcal{A}$ has a longest finite chain of subgroups. (Thus presheaves on $\mathbb{Z}$ are not graduated.) Presheaves on a poset $\mathcal{A}$ are graduated iff the down-set of every element is finite. Finally, presheaves on a category with binary products are graduated iff on every object there exist only finitely many sieves. Example: let $\mathcal{F}$ be the category of finite sets and mappings. Then presheaves on $\mathcal{F}^{op}$ (which form the category of all finitary sets functors) are graduated.

References

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