On preorders of path-complete Lyapunov functions through lifts

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Abstract

We consider a discrete-time switched system of the form

$$x(k+1) = f_{\sigma(k)}(x(k)), \quad k \in \mathbb{N}_{\geq 0},\, x(0) = x_0, \qquad (1)$$

where $\sigma : \mathbb{N}_{\geq 0} \to \Sigma$ is a switching signal, for Σ a finite set that parametrizes a set of continuous maps $F := \{f_i \in \mathscr{F} : i \in \Sigma\}$ on $\mathbb{R}^n$. We assume to have no knowledge of σ, and study stability of $0 \in \mathbb{R}^n$, uniform over all switching sequences. To do so, we elaborate on the framework of *path-complete Lyapunov functions* (PCLFs) [1], a framework that aims to unify and generalize the study of multiple Lyapunov functions.

A directed graph $\mathscr{G} = (S,E)$ with labels in Σ is said to be *path-complete*, denoted by $\mathscr{G} \in \mathrm{pc}(\Sigma)$, when for any (switching) sequence $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k) \in \Sigma^k$ there is a sequence of labeled edges $\{(s_j, s_{j+1}, \sigma(i_j))\}_{j=1,\dots,k}$ such that $(s_j, s_{j+1}, \sigma(i_j)) \in E$ for $j = 1,\dots,k$. Now, to connect these directed graphs to stability, let $\mathscr{V}$ be a template of candidate Lyapunov functions. Then, a *path-complete Lyapunov function* (PCLF) for F is the pair $(\mathscr{G} = (S,E), V_S)$ where $\mathscr{G} \in \mathrm{pc}(\Sigma)$ and $V_S = \{V_s : V_s \in \mathscr{V}, s \in S\}$ is a set of Lyapunov functions such that for all $(a,b,i) \in E$ we have that $V_a(x) \geq V_b(f_i(x))\ \forall x \in \mathbb{R}^n$. If this holds true, we denote this by $V_S \in \mathrm{pclf}(\mathscr{G}, F; \Sigma)$. A PCLF is in particular a sufficient condition for stability of 0 under (1), *e.g.*, see [1].

Selecting multiple Lyapunov functions, either graphical or not, remains an art and towards streamlining this study, we consider the following. Given $\mathscr{G}_1 = (S_1, E_1), \mathscr{G}_2 = (S_2, E_2) \in \mathrm{pc}(\Sigma)$, a template of Lyapunov functions $\mathscr{V}$ and a family of maps $\mathscr{F}$, if for any $F \in \mathscr{F}^{|\Sigma|}$ $\exists V_{S_1} \in \mathscr{V}^{|S_1|} : V_{S_1} \in \mathrm{pclf}(\mathscr{G}_1, F; \Sigma) \implies \exists V_{S_2} \in \mathscr{V}^{|S_2|} : V_{S_2} \in \mathrm{pclf}(\mathscr{G}_2, F; \Sigma)$, then

$$\mathscr{G}_1 \leq_{\mathscr{V},\mathscr{F}} \mathscr{G}_2. \qquad (2)$$

If (2) must hold for *any* $\mathscr{V}$ and $\mathscr{F}$ we write $\mathscr{G}_1 \leq \mathscr{G}_2$. One can interpret (2) as $\mathscr{G}_2$ being "*more expressive*" than $\mathscr{G}_1$ and in that sense $\mathscr{G}_2$ is understood as being "*better*". One concrete reason of interest is that this *preorder* has direct ramifications for approximating the joint spectral radius (JSR) in case $\mathscr{F}$ is the set of linear maps [1].

Unfortunately, (2) seems hopelessly intractable. To address this, we need one more notion. A graph $\mathscr{G}_1 = (S_1, E_1)$ is said to *simulate* $\mathscr{G}_2 = (S_2, E_2)$ when there is a map $R : S_2 \to S_1$ such that for all $(a,b,i) \in E_2$ we have that $(R(a), R(b), i) \in E_1$. Now, it is known that $\mathscr{G}_1$ simulating $\mathscr{G}_2$ is equivalent to $\mathscr{G}_1 \leq \mathscr{G}_2$ [3, Thm. 3.5]. Evidently, $\mathscr{G}_1 \leq \mathscr{G}_2$ is significantly stronger than $\mathscr{G}_1 \leq_{\mathscr{V},\mathscr{F}} \mathscr{G}_2$ and thus the naïve simulation relation fails to capture all these other preorders.

A way forward is to use so-called "*lifts*". For instance, let $\mathscr{G}_1, \mathscr{G}_2 \in \mathrm{pc}(\Sigma)$, then, the following are equivalent: (i) the *sum lift* of $\mathscr{G}_1$ simulates $\mathscr{G}_2$; and (ii) $\mathscr{G}_1 \leq_{\mathscr{V}} \mathscr{G}_2$ for any template $\mathscr{V}$ closed under addition [2, Thm. 1], that is, the *algebraic* statement $\mathscr{G}_1 \leq_{\mathscr{V}} \mathscr{G}_2$ admits a purely *combinatorial* reformulation via (i). Results of this form are scarce and a general understanding is lacking. In particular, the preorder induced by quadratic forms and linear maps is not yet understood, *e.g.*, although the set of quadratic forms is closed under addition, we can show that the sum lift provably fails to capture this preorder. Then, does a suitable lift exist?

For *any* pair $(\mathscr{V}, \mathscr{F})$, we aim to find a lift $L : \mathrm{pc}(\Sigma) \to \mathrm{pc}(\Sigma)$ such that $\forall \mathscr{G}_1, \mathscr{G}_2 \in \mathrm{pc}(\Sigma)$ the following are equivalent:

(I) $L(\mathscr{G}_1)$ simulates $\mathscr{G}_2$; and

(II) $\mathscr{G}_1 \leq_{\mathscr{V},\mathscr{F}} \mathscr{G}_2$.

We can prove the following.

Theorem (Lifts always exist)**.** *For any template $\mathscr{V}$ and class of maps $\mathscr{F}$, there exists a map $L : \mathrm{pc}(\Sigma) \to \mathrm{pc}(\Sigma)$, continuous with respect to the Alexandroff topology induced by the preorder $\leq_{\mathscr{V},\mathscr{F}}$, such that (I) and (II) are equivalent.*

Additionally, by identifying (2) with an Alexandroff topology, we can fully characterize $\mathrm{pc}(\Sigma)$ topologically.

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