Plane geometry and convexity of polynomial stability regions

Didier HENRION

Michael ŠEBEK

LAAS-CNRS
Univ. Toulouse
France

Fac. Elec. Engr.
Czech Tech. Univ. Prague
Czech Republic

EIGOPT Workshop
Leuven, Belgium
June 2008
Given real matrices $A, B, C$, find real matrix $K$ such that

$$\max \text{ real eig } (A + BKC) < 0$$

Spectral abscissa, typically non-convex non-smooth

7 plane problems $K \in \mathbb{R}^2$ can be studied and solved graphically

Motivation: provide geometric insight into SOF design problems
Intriguing observation

6 out of 7 plane SOF problems seem to have a convex feasibility set.

Can we explain why?

Can we detect convexity?

Can we (re)formulate the problem using convex programming?

Problem formally stated during an AIM workshop in 2005
Algebraic problem statement

Parametrized polynomial

\[ p(s, k) = p_0(s) + k_1 p_1(s) + k_2 p_2(s) \]

where \( p_i(s) \in \mathbb{R}[s] \) are given polynomials of \( s \in \mathbb{C} \) and \( k \in \mathbb{R}^2 \) contains design parameters

What are conditions on \( k_1, k_2 \) such that
\( p(s, k) \) is (Hurwitz) stable
i.e. all its roots lie in open left half-plane?
Hermite stability condition

\[ p(s) \text{ stable } \iff H(p) \succ 0 \]

Hermite matrix = Lyapunov matrix = Bézoutian matrix
Bézoutian resultant

Let \( a(u) \), \( b(u) \) be polynomials of degree \( n \)

Bézoutian matrix \( B_u(a, b) \) is the symmetric matrix of size \( n \) with entries \( b_{ij} \) satisfying linear equations

\[
\frac{a(u)b(v) - a(v)b(u)}{v - u} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} u^{i-1} v^{j-1}
\]

Polynomial \( r_u(a, b) = \det B_u(a, b) \) is the resultant of \( a(u) \) and \( b(u) \) with respect to \( u \)

Eliminates variable \( u \) from system of equations \( a(u) = b(u) = 0 \)
Hermite matrix as a Bézoutian

Hermite matrix of $p(s)$ defined as Bézoutian matrix of real part and imaginary part of $p(j\omega)$:

$$
p_R(\omega^2) = \text{Re} \, p(j\omega)
$$
$$
\omega p_I(\omega^2) = \text{Im} \, p(j\omega)
$$

that is, $H(p) = B_\omega(p_R(\omega^2), \omega p_I(\omega^2))$, the variable to be eliminated being frequency $\omega$

Polynomial $p(s)$ is stable if and only if $H(p) \succ 0$

Matrix $H(p)$ is symmetric and quadratic in coefficients of $p(s)$
Example: SOF problem NN1

\[ p_0(s) = s(s^2 - 13), \quad p_1(s) = s(s - 5), \quad p_2(s) = s + 1, \]
\[ p_R(\omega^2) = -k_1 \omega^2 + k_2, \quad p_I(\omega^2) = -\omega^2 - 13 - 5k_1 + k_2 \]

Quadratic matrix inequality (QMI)

\[
H(k) = \begin{bmatrix}
  k_2(-13 - 5k_1 + k_2) & 0 & -k_2 \\
  0 & k_1(-13 - 5k_1 + k_2) - k_2 & 0 \\
  -k_2 & 0 & k_1
\end{bmatrix} \succ 0
\]

equivalent to

\[
\begin{bmatrix}
  k_2(-13 - 5k_1 + k_2) & -k_2 \\
  -k_2 & k_1
\end{bmatrix} \succ 0, \quad k_1(-13 - 5k_1 + k_2) > 0
\]

In general, such QMIs define non-convex regions..

However here it is convex
Define stability region

\[ S = \{ k \in \mathbb{R}^2 : p(s, k) \text{ stable} \} \]

Is \( S \) convex?

If \( S \) is convex, give an LMI representation

\[ S = \{ k \in \mathbb{R}^2 : A_0 + k_1 A_1 + k_2 A_2 \succ 0 \} \]

when possible, where the \( A_i \) are real symmetric matrices to be found
Along the boundary

Define algebraic plane curve

\[ C = \{ k \in \mathbb{R}^2 : p(j\omega, k) = 0, w \in \mathbb{R} \} \]

which is the set of parameters \( k \) for which stability may be lost since \( s = j\omega \) sweeps the imaginary axis

Study of geometry of \( C \) = key idea of D-decomposition techniques (Neimark 1948)

Stability regions

Curve $\mathcal{C}$ partitions $\mathbb{C}$ into connected components $S_i, \ i = 1, 2\ldots$ within which the number of stable roots of $p$ remains constant.

Stability region $S$ corresponds to the union of components containing exactly $\deg p$ stable roots, hence

$$\partial S \subset \mathcal{C}$$

We study the geometry of $\mathcal{C}$ to understand the geometry of $S$. 
Elimination of frequency

Note that \( p(j\omega, k) = 0 \) for some \( \omega \in \mathbb{R} \) if and only if

\[
\begin{align*}
p_R(\omega^2, k) &= p_{0R}(\omega^2) + k_1p_{1R}(\omega^2) + k_2p_{2R}(\omega^2) = 0 \\
\omega p_I(\omega^2, k) &= \omega p_{0I}(\omega) + k_1\omega p_{1I}(\omega) + k_2\omega p_{2I}(\omega) = 0
\end{align*}
\]

Eliminate variable \( \omega \) via determinant of Hermite matrix

\[
h(k) = r_\omega(p_R(\omega^2, k), \omega p_I(\omega^2, k)) = \det H(k)
\]

Implicit algebraic description

\[
C = \{ k : h(k) = 0 \}
\]
Resultant factorisation

Resultant can be factored as

\[ h(k) = l(k)g(k)^2 \]

with \( l(k) \) linear, hence

\[ \mathcal{C} = \mathcal{L} \cup \mathcal{G} = \{ k : l(k) = 0 \} \cup \{ k : g(k) = 0 \} \]

Equation of line \( \mathcal{L} \)

\[ l(k) = r_\omega(p_R(\omega), \omega) = p_R(0, k) = p_0R(0) + k_1p_1R(0) + k_2p_2R(0) \]

Defining polynomial of the other curve component \( \mathcal{G} \)

\[ g(k) = r_\omega(p_R(\omega, k), p_I(\omega, k)) \]
Parametrising the other curve component

From relations
\[
\begin{bmatrix}
  p_1 R(\omega^2) & p_2 R(\omega^2) \\
  \omega p_1 I(\omega^2) & \omega p_2 I(\omega^2)
\end{bmatrix}
\begin{bmatrix}
  k_1 \\
  k_2
\end{bmatrix}
= -\begin{bmatrix}
  p_0 R(\omega^2) \\
  \omega p_0 I(\omega^2)
\end{bmatrix}
\]

we derive a rational parametrisation of \( G \)

\[
\begin{bmatrix}
  k_1(\omega^2) \\
  k_2(\omega^2)
\end{bmatrix}
= \frac{1}{p_1 I(\omega^2)p_2 R(\omega^2) - p_1 R(\omega^2)p_2 I(\omega^2)}
\begin{bmatrix}
  p_2 I(\omega^2) & -p_2 R(\omega^2) \\
  -p_1 I(\omega^2) & p_1 R(\omega^2)
\end{bmatrix}
\begin{bmatrix}
  p_0 R(\omega^2) \\
  \omega p_0 I(\omega^2)
\end{bmatrix}
= \begin{bmatrix}
  \frac{q_1(\omega^2)}{q_0(\omega^2)} \\
  \frac{q_2(\omega^2)}{q_0(\omega^2)}
\end{bmatrix}
\]

Using Bézoutians we can prove that symmetric pencil
\[ G(k) = B_\omega(q_1, q_2) + k_1 B_\omega(q_2, q_0) + k_2 B_\omega(q_1, q_0) \]
is such that
\[ G = \{ k : \text{det} G(k) = 0 \} \]
Determinantal locus

Defining $C(k) = \text{diag} \{ l(k), G(k) \}$, algebraic curve $C$ can be described as a determinantal locus

$$C = \{ k : \det C(k) = 0 \}$$

with $C(k)$ symmetric linear in $k$

Recall that $C$ partitions $\mathbb{C}$ into connected components $S_i$

If $C(k) \succ 0$ for some point $k$ in the interior of $S_i$ for some $i$, then $S_i = \{ k : C(k) \succeq 0 \}$ is a convex LMI region

Converse is false: $S_i$ may be convex, but not LMI.
Rigid convexity

Convex sets which admit an LMI representation are called **rigidly convex** by Helton and Vinnikov (2002)

Rigid convexity is stronger than convexity

Algebraic characterisation: a generic line through the interior should intersect Zariski closure of the boundary at real points

Equivalent to polynomial hyperbolicity (barrier functions, PDEs)
TV screen or Fermat quartic
Bivariate polynomial of degree 8
Trivariate polynomial of degree 3 (Cayley cubic)
LMI stability regions

It may happen that

- $S_i$ is convex for some $i$, yet $C(k)$ is not positive definite for points $k$ within $S_i$
- $S_i$ is rigidly convex (≡ convex LMI) for some $i$, yet $H(k)$ is not positive definite for points $k$ within $S_i$

If $H(k) \succ 0$ and $C(k) \succ 0$ for some point $k$ in the interior of $S_i$, then $S_i$ is an LMI region included in stability region $S$

Quite often, on practical instances, we observe that

\[ S = S_i \]

for some $i$ and moreover $S_i$ is convex LMI
Hermite matrix

\[
H(k) = \begin{bmatrix}
  k_2(-13 - 5k_1 + k_2) & 0 & -k_2 \\
  0 & k_1(-13 - 5k_1 + k_2) - k_2 & 0 \\
  -k_2 & 0 & k_1
\end{bmatrix}
\]

Hence \( h(k) = \det H(k) = k_2(-13k_1 - k_2 - 5k_1^2 + k_1k_2)^2 \) and then
\( l(k) = k_2, \ g(k) = -13k_1 - k_2 - 5k_1^2 + k_1k_2 \)

Rational parametrization of curve \( G = \{ k : g(k) = 0 \} \) given by

\[
\begin{align*}
  k_1(\omega^2) &= (\omega^2 + 13)/(\omega^2 - 5) \\
  k_2(\omega^2) &= \omega^2(\omega^2 + 13)/(\omega^2 - 5)
\end{align*}
\]

obtained with \texttt{algcurves} package of Maple
Symmetric affine determinantal representation of $G = \{k : \det G(k) = 0\}$ given by

$$G(k) = \begin{bmatrix} 169 + 65k_1 - 18k_2 & 13 + 5k_1 \\ 13 + 5k_1 & 1 - k_1 \end{bmatrix}$$

obtained via LinearAlgebra[BezoutMatrix] function of Maple.

Symmetric affine determinantal representation of $C = \{k : \det C(k) = 0\}$ given by

$$C(k) = \begin{bmatrix} k_2 & 0 & 0 \\ 0 & 169 + 65k_1 - 18k_2 & 13 + 5k_1 \\ 0 & 13 + 5k_1 & 1 - k_1 \end{bmatrix}$$

LMI stability region

$$S = \{k : C(k) \succ 0\}$$
Example 14.4 from Ackermann (1993)

\[ p_0(s) = s^4 + 2s^3 + 10s^2 + 10s + 14 + 2a, \]
\[ p_1(s) = 2s^3 + 2s - 3/10, \]
\[ p_2(s) = 2s + 1, \text{ with parameter } a \in \mathbb{R} \]

We obtain \( C(k) = \text{diag} \{ l(k), G(k) \} \) with

\[
\begin{align*}
l(k) & = 140 + 20a - 3k_1 + 10k_2 \\
G_{11}(k) & = 7920 + 4860a + 400a^2 + (-1609 - 60a)k_1 \\
& \quad + (-270 + 200a)k_2 \\
G_{21}(k) & = -8350 - 2000a + 1430k_1 + 130k_2 \\
G_{22}(k) & = 8370 - 1230k_1 - 100k_2 \\
G_{31}(k) & = 900 + 200a - 130k_1 \\
G_{32}(k) & = -900 + 100k_1 \\
G_{33}(k) & = 100
\end{align*}
\]
When $a = 1$, $S$ consists of two disconnected regions and the region including the origin is LMI.
When $a = 0$, LMI region $\{k : C(k) \succ 0\}$ is not included in non-convex $S$. 
Conclusion

Algebraic geometry explains why some plane stability regions can be convex.

Allows to detect convexity and find explicit LMI representations.

Relies on algebraic plane curve parametrizations and Bézoutians, hence extension to more than 2 parameters seems to be difficult.

Results not useful for optimisation purposes ($H_\infty$ etc) since bivariate LMIs can be solved algebraically.
Determinantal representation

Key issue: finding a symmetric affine determinantal representation of multivariate polynomials

Very difficult in general, should exploit rational parametrisation of the corresponding hypersurface

Simplest 3rd degree case $p(s, k) = s^3 + k_1 s^2 + k_2 s + k_3$

Find four symmetric real matrices $A_0, A_1, A_2, A_3$ such that

$$\det(A_0 + A_1 k_1 + A_2 k_2 + A_3 k_3) = k_1 k_3 - k_2$$