Introduction: An average consensus problem

- Agents move in discrete time
- They average their neighbours’ positions
- They converge to average consensus
A rogue agent takes power

- One agent changes its connections
- End position: far from average
- Conditions for perturbation to preserve (close to) average?
- Enough to stay connected?
Opinion dynamics and dictators

- Political opinion: number in [-1,1]
- People talk with friends and change their opinion
- Same dynamics as consensus
- How can someone impose his opinion to everybody else?
- How to preserve democracy?
Consensus = opinion dynamics = Markov chains

Consensus: $x_{t+1} = Ax_t$, A stochastic

Opinion dynamics: $x_{t+1} = Ax_t$, A stochastic

Typically: A sparse = small outdegree

Markov chains: $\pi_{t+1} = \pi_tA$

Entries of $\pi = \text{influence of agents in final decision}$
Democracy in Markov chains and its preservation under local perturbations

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This is not classical perturbation theory

- Matrix A is
  - stochastic
  - sparse (few entries per row)
  - large
- Left dominant eigenvector $\pi$
- Classical perturbation theory:
  If we change all entries by a small $\varepsilon$, what happens to $\pi$?
This is ‘combinatorial perturbation’ theory

- Matrix A is
  - stochastic
  - sparse (few entries per row)
  - large
- Left dominant eigenvector $\pi$
- Combinatorial perturbation theory:
  If we change a few entries by any amount, what happens to $\pi$?
Or rather: ‘asymptotic’ combinatorial perturbation theory

- We wish we had a bound on $\|\pi - \pi'\|$ depending on number of entries changed, dimension of $A$, etc.
- Hard!
- Simpler:
  - Large $\rightarrow$ Larger and larger
  - Few $\rightarrow$ bounded
  - What happens to $\|\pi\|, \|\pi'\|, \|\pi - \pi'\|$?
Sequences of chains and democracy

- Set of nodes $V_1 \subset V_2 \subset V_3 \ldots \subset V_\infty$
- Mixing Markov chain $G_n$ on $V_n$
- $G_n$ stabilises when $n \to 1$
  For all nodes $x,y$: $G_n(x,y)$ eventually constant
- $\pi_n = $ stationary distribution of $G_n$
- $\|\pi_n\|_\infty = $ largest entry of $\pi_n$
Democracy in Markov chains

- **Democratic** iff $\|\pi_n\|_\infty \rightarrow 0$

- Democracy for opinion/consensus
  - = no dictator
  - = no agent has a dominant influence on the final opinion/consensus

- Similar to ‘wisdom of crowds’ (M.O. Jackson)
The ring is democratic

$G_n$:

$G_1$:
Reversible random walks

- $G_n$: random walk on a connected undirected graph
- $\pi_n$: normalised degrees
- If bounded degree, then democratic

- The infinite graph can be weighted
  - If weights bounded from above and below, and bounded degree, then democratic

- All those examples are reversible
- In fact, general for reversible chains
Finite perturbation

- Finite set of nodes $S$
- In all $G_n$, rewire all edges from $S$ arbitrarily (independently of $n$)
- We assume the rewired $G'_n$ is still mixing

- Is democracy preserved? Not always.
Example: biased ring

\[ G_n : \]

\[ 1 \quad 0.2 \quad 0.2 \quad 0.8 \quad 0.8 \quad 0.2 \]

\[ 2 \quad 0.2 \quad 0.8 \quad 0.8 \quad 0.2 \]

\[ 3 \quad 0.8 \quad 0.2 \quad 0.8 \quad 0.2 \]

\[ 4 \quad 0.2 \quad 0.8 \quad 0.8 \quad 0.2 \]

\[ 5 \quad 0.8 \quad 0.2 \quad 0.8 \quad 0.2 \]

\[ \ldots \]
The perturbed biased ring

$G'_n$:

$\pi_n(1)$ converges to a non zero value

) Not democratic
The perturbed biased ring

Limit chain: not connected!

Is it enough to ask for a irreducible limit?
Theorem:
A sequence of random walks on undirected, bounded-degree graphs with weights from a finite set
- is democratic
- remains so under finite perturbations which preserve irreducibility of the limit chain.

The perturbed chains are not necessarily reversible anymore.
Tools

- $\pi^{-1}(x) =$ first return time of node $x$

  democracy $=$ smallest first return time grows to infinity

- We can permute the nodes and converge to a different limit chain.

Look at all limit chains: do they have a stationary distribution?
Conclusions

- ‘Combinatorial perturbation’ of Markov chains/consensus
- Democracy = every agent has negligible influence
- Robust to local irreducibility-preserving perturbations for reversible, bounded-degree random walks
- Open problems:
  - Lift assumptions
  - Slowly growing set of rogue agents
  - Finitary versions
  - Different norms
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