Some results on the stabilization and on the controllability of nonlinear wave equations

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Outline

Stabilization: Dissipative boundary conditions for hyperbolic equations

- The equations
- Main result
- Comparison with prior results
- Proof of the exponential stability
- Application to the control of open channels
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  The equations
  Main result
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Controllability of hyperbolic systems
  The control problem
  Controllability theorem
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Controllability of the Korteweg-de Vries (KdV) equation
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La Sambre
Commercial break
Control and Nonlinearity

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- Assumptions on \( A \)

\[ A(0) = \text{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n), \quad \Lambda_i > 0, \; \Lambda_i < 0, \; \forall i \in \{1, \ldots, m\}, \quad \Lambda_i \neq \Lambda_j, \; \forall (i, j) \in \{1, \ldots, n\}^2 \text{ such that } i \neq j. \]
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• Boundary conditions on $y$:

\[
\begin{pmatrix}
    y_+(t, 0)
    \\
    y_-(t, 1)
\end{pmatrix}
= G
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    \\
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(i) $y_+ \in \mathbb{R}^m$ and $y_- \in \mathbb{R}^{n-m}$ are defined by

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y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix},
$$

(ii) the map $G : \mathbb{R}^n \to \mathbb{R}^n$ vanishes at 0.
Notations

For $K \in \mathcal{M}_{n,m}(\mathbb{R})$, 

$$∥K∥ := \max\{|Kx|; x \in \mathbb{R}^n, |x| = 1\}.$$
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If $n = m$, 

$$\rho_1(K) := \inf \{\|\Delta K \Delta^{-1}\|; \Delta \in \mathcal{D}_{n,+}\},$$ 

where $\mathcal{D}_{n,+}$ denotes the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements.
Theorem 1.1 (JMC-G. Bastin-B. d’Andréa-Novel (2008)).

If \( \rho_1(G'(0)) < 1 \), then the equilibrium \( \bar{y} \equiv 0 \) of the quasi-linear hyperbolic system

\[
y_t + A(y)y_x = 0,
\]

with the above boundary conditions, is exponentially stable for the Sobolev \( H^2 \)-norm.
Estimate on the exponential decay rate

For every $\nu \in (0, -\min\{|\Lambda_1|, \ldots, |\Lambda_n|\} \ln(\rho_1(G'(0))))$, there exist $\varepsilon > 0$ and $C > 0$ such that, for every $y_0 \in H^2((0, 1), \mathbb{R}^n)$ satisfying $|y_0|_{H^2((0,1),\mathbb{R}^n)} < \varepsilon$ (and the usual compatibility conditions) the classical solution $y$ to the Cauchy problem

$$y_t + A(y)y_x = 0, \quad y(0, x) = y_0(x) + \text{boundary conditions}$$

is defined on $[0, +\infty)$ and satisfies

$$|y(t, \cdot)|_{H^2((0,1),\mathbb{R}^n)} \leq Ce^{-\nu t}|y_0|_{H^2((0,1),\mathbb{R}^n)}, \quad \forall t \in [0, +\infty).$$
The Ta-tsien Li condition

\[ R_2(K) := \text{Max} \left\{ \sum_{j=1}^{n} |K_{ij}|; \; i \in \{1, \ldots, n\} \right\}, \]

\[ \rho_2(K) := \text{Inf} \left\{ R_2(\Delta K \Delta^{-1}); \; \Delta \in \mathcal{D}_{n,+} \right\}. \]

**Theorem 1.2 (Ta-tsien Li (1994)).**

If \( \rho_2(G'(0)) < 1 \), then the equilibrium \( \bar{y} \equiv 0 \) of the quasi-linear hyperbolic system

\[ y_t + A(y)y_x = 0, \]

with the above boundary conditions, is exponentially stable for the \( C^1 \)-norm.
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$C^1/H^2$-exponential stability

1. Open problem: Does there exist $K$ such that one has exponential stability for the $C^1$-norm but not for the $H^2$-norm?
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Comparison of $\rho_2$ and $\rho_1$

Proposition 1.3. For every $K \in M_{n \times n}(\mathbb{R})$, $\rho_1(K) \leq \rho_2(K)$.

(3)

Example where (3) is strict: for $a > 0$, let $K_a := \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in M_{2 \times 2}(\mathbb{R})$. Then $\rho_1(K_a) = \sqrt{2}a < 2a = \rho_2(K_a)$.

Open problem: Does $\rho_1(K) < 1$ implies the exponential stability for the $C_1$-norm?
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Comparison with stability conditions for linear hyperbolic systems

For simplicity we assume that $\Lambda_i$ are all positive and consider we consider the special case of linear hyperbolic systems

$$y_t + \Lambda y_x = 0, \quad y(t, 0) = Ky(t, 1),$$

where

$$\Lambda := \text{diag} (\Lambda_1, \ldots, \Lambda_n), \quad \text{with } \Lambda_i > 0, \quad \forall i \in \{1, \ldots, n\}.$$ 

**Theorem 1.4.**

Exponential stability for the $C^1$-norm is equivalent to the exponential stability in the $H^2$-norm.
A Necessary and sufficient condition for exponential stability

Notation:

\[ r_i = \frac{1}{\Lambda_i}, \forall i \in \{1, \ldots, n\} \]

**Theorem 1.5.**
\( \bar{y} \equiv \) is exponentially stable for the system

\[ \dot{y} + \Lambda y = 0, \ y(t, 0) = Ky(t, 1) \]

if and only if there exists \( \delta > 0 \) such that

\[ \left( \det \left( \text{Id}_n - \left( \text{diag} \left( e^{-r_1 z}, \ldots, e^{-r_n z} \right) \right) K \right) = 0, \ z \in \mathbb{C} \right) \Rightarrow (\Re(z) \leq -\delta). \]
An example

Let us choose $\lambda_1 := 1$, $\lambda_2 := 2$ (hence $r_1 = 1$ and $r_2 = 1/2$ and

$$K_a := \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \ a \in \mathbb{R}.$$ 

Then $\rho_1(K) = 2|a|$. Hence $\rho_1(K_a) < 1$ is equivalent to $a \in (-1/2, 1/2)$. 

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Then $\rho_1(K) = 2|a|$. Hence $\rho_1(K_a) < 1$ is equivalent to $a \in (-1/2, 1/2)$. However exponential stability is equivalent to $a \in (-1, 1/2)$. 
Robustness issues

For a positive integer $n$, let

$$
\Lambda_1 := \frac{4n}{4n+1}, \quad \Lambda_2 = \frac{4n}{2n+1}.
$$

Then

$$
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} := \begin{pmatrix}
\sin \left(4n\pi \left(t - \frac{x}{\Lambda_1}\right)\right) \\
\sin \left(4n\pi \left(t - \frac{x}{\Lambda_2}\right)\right)
\end{pmatrix}
$$

is a solution of $y_t + \Lambda y_x$, $y(t, 0) = K_{-1/2}y(t, 1)$ which does not tends to 0 as $t \to +\infty$. Hence one does not have exponential stability. However $\lim_{n \to +\infty} \Lambda_1 = 1$ and $\lim_{n \to +\infty} \Lambda_2 = 2$. 

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Robust exponential stability

Notations:

\[ \rho_0(K) := \max\{ \rho(\text{diag} (e^{i\theta_1}, \ldots, e^{i\theta_n}) K); (\theta_1, \ldots, \theta_n)^{\text{tr}} \in \mathbb{R}^n \} \]

Theorem 1.6 (R. Silkowski (1993)).

If the \((r_1, \ldots, r_n)\) are rationally independent, the linear system
\[
y_t + \Lambda y_x = 0, \quad y(t, 0) = Ky(t, 1)
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is exponentially stable if and only if \(\rho_0(K) < 1\).
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**Theorem 1.6 (R. Silkowski (1993)).**

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Note that \( \rho_0(K) \) depends continuously on \( K \) and that “\((r_1, \ldots, r_n)\) are rationally independent” is a generic condition. Therefore, if one wants to have a natural robustness property with respect to the \( r_i \)'s, the condition for exponential stability is

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*This condition does not depend on the \(\Lambda_i\)’s!*
Comparison of $\rho_0$ and $\rho_1$

Proposition 1.7 (JMC-G. Bastin-B. d’Andrea-Novel (2008)).

For every $n \in \mathbb{N}$ and for every $K \in M_n(R)$, $\rho_0(K) \leq \rho_1(K)$.

For every $n \in \{1, 2, 3, 4, 5\}$ and for every $K \in M_n(R)$, $\rho_0(K) = \rho_1(K)$.

For every $n \in \mathbb{N} \setminus \{1, 2, 3, 4, 5\}$, there exists $K \in M_n(R)$ such that $\rho_0(K) < \rho_1(K)$.

Open problem: Is $\rho_0(G'(0)) < 1$ a sufficient condition for exponential stability (for the $H_2$-norm) in the nonlinear case?
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Proof of the exponential stability if $A$ is constant and $G$ is linear

Main tool: a Lyapunov approach. $A(y) = \Lambda$, $G(y) = Ky$. For simplicity, all the $\Lambda_i$'s are positive. Lyapunov function candidate:

$$V(y) := \int_0^1 y^{\text{tr}} Q ye^{-\mu x} dx, \quad Q \text{ is positive symmetric}.$$
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$$V(y) := \int_0^1 y^{tr} Q y e^{-\mu x} dx, \quad Q \text{ is positive symmetric.}$$

$$\dot{V} = -\int_0^1 (y_x^{tr} \Lambda Q y + y^{tr} Q y_x) \Lambda e^{-\mu x} dx$$

$$= -\mu \int_0^1 y^{tr} \Lambda Q y \ e^{-\mu x} dx - B,$$

with

$$B := [y^{tr} \Lambda Q y e^{-\mu x}]_{x=1}^{x=0} = y(1)^{tr} (\Lambda Q e^{-\mu} - K^{tr} \Lambda Q K)y(1)$$
Let $D \in \mathcal{D}_{n,+}$ be such that $\|D KD^{-1}\| < 1$ and let $\xi := Dy(1)$. We take $Q = D^2 \Lambda^{-1}$. Then

$$B = e^{-\mu} |\xi|^2 - |DKD^{-1}\xi|^2.$$ 

Therefore it suffices to take $\mu > 0$ small enough.

**Remark 1.8.**

*Introduction of $\mu$:*
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- JMC (1998) for the stabilization of the Euler equations.
New difficulties if $A(y)$ depends on $y$

We try with the same $V$:

$$
\dot{V} = - \int_0^1 (y_x^\text{tr} A(y)^{\text{tr}} Q y + y^\text{tr} Q A(y) y_x) e^{-\mu x} \, dx
$$

$$
= -\mu \int_0^1 y^\text{tr} A(y) Q y e^{-\mu x} \, dx - B + N_1 + N_2
$$

with

$$
N_1 := \int_0^1 y^\text{tr} (Q A(y) - A(y) Q) y_x e^{-\mu x} \, dx,
$$

$$
N_2 := \int_0^1 y^\text{tr} (A'(y) y_x)^{\text{tr}} Q y e^{-\mu x} \, dx
$$
Solution for $N_1$

Take $Q$ depending on $y$ such that $A(y)Q(y) = Q(y)A(y)$, $Q(0) = D^2 F(0)^{-1}$. (This is possible since the eigenvalues of $F(0)$ are distinct.) Now

$$\dot{V} = -\mu \int_0^1 y^{tr} A(y)Q(y) ye^{-\mu x} \, dx - B + N_2$$

with

$$N_2 := \int_0^1 y^{tr} (A'(y)y_x Q(y) + A(y)Q'(y)y_x)^{tr} ye^{-\mu x} \, dx$$
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What to do with $N_2$?
Solution for $N_2$

New Lyapunov function:

$$V(y) = V_1(y) + V_2(y) + V_3(y)$$

with

$$V_1(y) = \int_0^1 y^{\text{tr}} Q(y) y \ e^{-\mu x} \ dx,$$

$$V_2(y) = \int_0^1 y_x^{\text{tr}} R(y) y_x \ e^{-\mu x} \ dx,$$

$$V_3(y) = \int_0^1 y_{xx}^{\text{tr}} S(y) y_{xx} \ e^{-\mu x} \ dx,$$

where $\mu > 0$, $Q(y)$, $R(y)$ and $S(y)$ are symmetric positive definite matrices.
Choice of $Q$, $R$ and $S$

- Commutations:

\[
A(y)Q(y) - Q(y)A(y) = 0,
A(y)R(y) - R(y)A(y) = 0,
A(y)S(y) - S(y)A(y) = 0,
\]
Choice of $Q$, $R$ and $S$

- Commutations:

\[ A(y)Q(y) - Q(y)A(y) = 0, \]
\[ A(y)R(y) - R(y)A(y) = 0, \]
\[ A(y)S(y) - S(y)A(y) = 0, \]

- \[
Q(0) = D^2 A(0)^{-1}, \quad R(0) = D^2 A(0), \quad S(0) = D^2 A(0)^3.
\]
Lemma 1.9.

If $\mu > 0$ is small enough, there exist positive real constants $\alpha$, $\beta$, $\delta$ such that, for every $y : [0, 1] \to \mathbb{R}^n$ such that $|y|_{C^0([0,1])} + |y_x|_{C^0([0,1])} \leq \delta$, we have

$$\frac{1}{\beta} \int_0^1 (|y|^2 + |y_x|^2 + |y_{xx}|^2)dx \leq V(y) \leq \beta \int_0^1 (|y|^2 + |y_x|^2 + |y_{xx}|^2)dx,$$

$$\dot{V} \leq -\alpha V.$$

...
La Sambre (G. Bastin, L. Moens, ... )
The Saint-Venant equations

The index $i$ is for the $i$-th reach.
Conservation of mass:

$$H_{it} + (H_i V_i)_x = 0,$$
The Saint-Venant equations

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Conservation of momentum:

$$V_{it} + \left( gH_i + \frac{V_i^2}{2} \right)_x = 0.$$
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Flow rate: $Q_i = H_i V_i$. 
Barré de Saint-Venant
(Adhémar-Jean-Claude)
1797-1886
Barré de Saint-Venant
(Adhémar-Jean-Claude)
1797-1886
Boundary conditions

Underflow (sluice)

\[ Q = K \sqrt{\beta (H_{up} - H_{down})} \]

Overflow (spillway)

\[ Q = K (H_{up} - \beta)^{3/2} \]
La Sambre: Hydraulic gates
Closed loop versus open loop
Work in progress: La Meuse
Complements


• Spectral methods: X. Litrico and V. Fromion (2006).


Stabilization: Dissipative boundary conditions for hyperbolic equations

The equations
Main result
Comparaison with prior results
Proof of the exponential stability
Application to the control of open channels

Controllability of hyperbolic systems

The control problem
Controllability theorem

Controllability of the Korteweg-de Vries (KdV) equation
Local controllability
Global controllability
The control problem

Let $T > 0$. Given $y^0 : [0, 1] \rightarrow \mathbb{R}^n$ and $y^1 : [0, 1] \rightarrow \mathbb{R}^n$. Does there exist $u_+ : [0, T] \rightarrow \mathbb{R}^m$, $u_- : [0, T] \rightarrow \mathbb{R}^m$ such that the solution of the Cauchy problem

$$y_t + A(y)y_x = 0, \quad y_+(t, 0) = u_+(t), \quad y_-(t, 0) = u_-(t), \quad y(0, x) = y^0(x),$$

satisfies $y(T, x) = y^T(x)$?
The control problem

Let $T > 0$. Given $y^0 : [0, 1] \rightarrow \mathbb{R}^n$ and $y^1 : [0, 1] \rightarrow \mathbb{R}^n$. Does there exist $u_+ : [0, T] \rightarrow \mathbb{R}^m$, $u_- : [0, T] \rightarrow \mathbb{R}^m$ such that the solution of the Cauchy problem

$$y_t + A(y)y_x = 0, \quad y_+(t, 0) = u_+(t), \quad y_-(t, 0) = u_-(t), \quad y(0, x) = y^0(x),$$

satisfies $y(T, x) = y^T(x)$?

Local controllability: $y^0$ and $y^T$ are small.

**Remark 2.1.**

*The control is on both sides. The case where the control is only on one side can also be considered.*
Controllability theorem

**Theorem 2.2 (Ta-tsien Li and Bopeng Rao (2003)).**

*(Local controllability for the $C^1$-norm)*

\[
T > \max \left\{ \frac{1}{\Lambda_1}, \ldots, \frac{1}{\Lambda_m}, \frac{1}{|\Lambda_{m+1}|}, \ldots, \frac{1}{|\Lambda_n|} \right\}
\]

**Remark 2.3.**

• If $T > \max \left\{ \frac{1}{\Lambda_1}, \ldots, \frac{1}{\Lambda_m}, \frac{1}{|\Lambda_{m+1}|}, \ldots, \frac{1}{|\Lambda_n|} \right\}$, the linearized control system at $(\bar{y}, \bar{u}) = (0, 0)$ is controllable.

• One of the main ingredients of the proof: consider $y_t + A(y)x = 0$ as an evolution equation in $x$:

\[
A(y) - \frac{1}{y_t} + y_x = 0.
\]

• Generalization: $A(t, x, y)$: Zhiqiang Wang (2007).

• Applications to channels: M. Gugat and G. Leugering (2008).

• For the control on one side only, see Ta-tsien Li and Bopeng Rao (2002).
Theorem 2.2 (Ta-tsien Li and Bopeng Rao (2003)).

(Local controllability for the $C^1$-norm)

\[ \iff \left( T > \max \left\{ \frac{1}{\Lambda_1}, \ldots, \frac{1}{\Lambda_m}, \frac{1}{|\Lambda_{m+1}|}, \ldots, \frac{1}{|\Lambda_n|} \right\} \right) \]

Remark 2.3.

- If $T > \max \left\{ \frac{1}{\Lambda_1}, \ldots, \frac{1}{\Lambda_m}, \frac{1}{|\Lambda_{m+1}|}, \ldots, \frac{1}{|\Lambda_n|} \right\}$, the linearized control system at $(\bar{y}, \bar{u}) = (0, 0)$ is controllable.

- One of the main ingredients of the proof: consider $y_t + A(y)y_x = 0$ as an evolution equation in $x$: $A(y)^{-1}y_t + y_x = 0$.


- For the control on one side only, see Ta-tsien Li and Bopeng Rao (2002).
Stabilization: Dissipative boundary conditions for hyperbolic equations

- The equations
- Main result
- Comparison with prior results
- Proof of the exponential stability
- Application to the control of open channels

Controllability of hyperbolic systems

- The control problem
- Controllability theorem

Controllability of the Korteweg-de Vries (KdV) equation

- Local controllability
- Global controllability
History of the KdV equation

- 1834 John Scott Russell
- 1872 Joseph Valentin de Boussinesq
- 1895 Diederik Korteweg and Gustav de Vries
- 1995 Experiment
The 1995 experiment
The KdV control system

\[ y_t + y_x + y_{xxx} + yy_x = 0, \ t \in [0, T], \ x \in [0, L], \]
\[ y(t, 0) = y(t, L) = 0, \ y_x(t, L) = u(t), \ t \in [0, T]. \]

where, at time \( t \in [0, T] \), the control is \( u \in \mathbb{R} \) and the state is \( y(t, \cdot) \in L^2(0, L) \).
Controllability of the linearized control system

The linearized control system (around 0) is

\[ y_t + y_x + y_{xxx} = 0, \; t \in [0, T], \; x \in [0, L], \]
\[ y(t, 0) = y(t, L) = 0, \; y_x(t, L) = u(t), \; t \in [0, T]. \]

where, at time \( t \in [0, T] \), the control is \( u \in \mathbb{R} \) and the state is \( y(t, \cdot) \in L^2(0, L) \).

**Theorem 3.1 (L. Rosier (1997)).**

For every \( T > 0 \), the linearized control system is controllable in time \( T \) if and only

\[ L \notin \mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} \right\}. \]
Proof of the “only if” part

$L$ is in $\mathcal{N}$ if and only if there exists $\varphi \neq 0 : [0, L] \to \mathbb{C}$ and $\lambda \in \mathbb{C}$ such that

$$\varphi_x + \varphi_{xxx} = \lambda \varphi, \quad \varphi(0) = \varphi_x(0) = \varphi(L) = \varphi_x(L) = 0.$$  

With such a $\varphi$, whatever is $u(t)$, the solution $y$ of the linearized control system satisfies

$$\frac{d}{dt} \int_0^L y \varphi = \lambda \int_0^L y \varphi$$

...
Connection with the Hautus criterion

Hautus criterion: in finite dimension $\dot{y} = Ay + Bu$ is controllable if and only if

$$(A^* \phi = \lambda \phi \text{ and } B^* \phi = 0) \Rightarrow (\phi = 0).$$

In infinite dimension the Hautus criterion is still necessary, but not sufficient in general. However, in infinite dimension, the Hautus criterion is sufficient provided that one has enough compactness. Here: T. Kato (1983) smoothing effect (for $x \in \mathbb{R}$; for $x \in [0, L]$: L. Rosier (1997)).
Theorem 3.2 (L. Rosier (1997)).

For every $T > 0$, the KdV control system is locally controllable (around 0) in time $T$ if $L \notin \mathcal{N}$. 
Application to the nonlinear system

**Theorem 3.2 (L. Rosier (1997)).**

For every $T > 0$, the KdV control system is locally controllable (around 0) in time $T$ if $L \not\in \mathcal{N}$.

Question: Does one have controllability if $L \in \mathcal{N}$?
Controllability when $L \in \mathcal{N}$

**Theorem 3.3 (JMC and E. Crépeau (2003)).**

If $L = 2\pi$ (which is in $\mathcal{N}$: take $k = l = 1$), for every $T > 0$ the KdV control system is locally controllable (around 0) in time $T$.

**Theorem 3.4 (E. Cerpa (2007), E. Cerpa and E. Crépeau (2008)).**

For every $L \in \mathcal{N}$, there exists $T > 0$ such that the KdV control system is locally controllable (around 0) in time $T$. 
Strategy of the proof: power series expansion.

Example with \( L = 2\pi \). For every trajectory \((y, u)\) of the linearized control system around 0

\[
\frac{d}{dt} \int_0^{2\pi} (1 - \cos(x)) y(t, x) dx = 0.
\]

This is the only “obstacle” to the controllability of the linearized control system:

**Proposition 3.5 (L. Rosier (1997)).**

Let \( H := \{ y \in L^2(0, L); \int_0^L (1 - \cos(x)) y(x) dx = 0 \} \). For every \((y^0, y^1) \in H \times H\), there exists \( u \in L^2(0, T) \) such that the solution to the Cauchy problem

\[
y_t + y_x + y_{xxx} = 0, \quad y(t, 0) = y(t, L) = 0, \quad y_x(t, L) = u(t), \quad t \in [0, T],
\]

\[
y(0, x) = y^0(x), \quad x \in [0, L],
\]

satisfies \( y(T, x) = y^1(x), \quad x \in [0, L] \).
We explain the method on the control system of finite dimension

\[
\dot{x} = f(x, u),
\]

where the state is \(x \in \mathbb{R}^n\) and the control is \(u \in \mathbb{R}^m\). We assume that \((0, 0) \in \mathbb{R}^n \times \mathbb{R}^m\) is an equilibrium of the control system (7), i.e \(f(0, 0) = 0\). Let

\[
H := \text{Span} \{A^i Bu; u \in \mathbb{R}^m, i \in \{0, \ldots, n-1\}\}
\]

with

\[
A := \frac{\partial f}{\partial x}(0, 0), \quad B := \frac{\partial f}{\partial u}(0, 0).
\]

If \(H = \mathbb{R}^n\), the linearized control system around \((0, 0)\) is controllable and therefore the nonlinear control system \(\dot{x} = f(x, u)\) is small-time locally controllable at \((0, 0) \in \mathbb{R}^n \times \mathbb{R}^m\).
Let us look at the case where the dimension of $H$ is $n-1$. Let us make a (formal) power series expansion of the control system $\dot{x} = f(x, u)$ in $(x, u)$ around 0. We write

\[ x = y^1 + y^2 + \ldots, \quad u = \nu^1 + \nu^2 + \ldots. \]

The order 1 is given by $(y^1, \nu^1)$; the order 2 is given by $(y^2, \nu^2)$ and so on. The dynamics of these different orders are given by

\[ \dot{y}^1 = \frac{\partial f}{\partial x}(0, 0)y^1 + \frac{\partial f}{\partial u}(0, 0)\nu^1, \]

\[ \dot{y}^2 = \frac{\partial f}{\partial x}(0, 0)y^2 + \frac{\partial f}{\partial u}(0, 0)\nu^2 + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0, 0)(y^1, y^1) \]
\[ + \frac{\partial^2 f}{\partial x \partial u}(0, 0)(y^1, \nu^1) + \frac{1}{2} \frac{\partial^2 f}{\partial u^2}(0, 0)(\nu^1, \nu^1), \]

and so on.
Let $e_1 \in H^\perp$. Let $T > 0$. Let us assume that there are controls $v^1_\pm$ and $v^2_\pm$, both in $L^\infty((0, T); \mathbb{R}^m)$, such that, if $y^1_\pm$ and $y^2_\pm$ are solutions of

$$
\dot{y}^1_\pm = \frac{\partial f}{\partial x}(0, 0)y^1_\pm + \frac{\partial f}{\partial u}(0, 0)v^1_\pm,
$$

$$
y^1_\pm(0) = 0,
$$

$$
\dot{y}^2_\pm = \frac{\partial f}{\partial x}(0, 0)y^2_\pm + \frac{\partial f}{\partial u}(0, 0)v^2_\pm + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0, 0)(y^1_\pm, y^1_\pm)
$$

$$
+ \frac{\partial^2 f}{\partial x \partial u}(0, 0)(y^1_\pm, u^1_\pm) + \frac{1}{2} \frac{\partial^2 f}{\partial u^2}(0, 0)(u^1_\pm, u^1_\pm),
$$

$$
y^2_\pm(0) = 0,
$$

then

$$
y^1_\pm(T) = 0,
$$

$$
y^2_\pm(T) = \pm e_1.
$$
Let \((e_i)_{i \in \{2, \ldots, n\}}\) be a basis of \(H\). By the definition of \(H\), there are \((u_i)_{i=2,\ldots,n}\), all in \(L^\infty(0, T)^m\), such that, if \((x_i)_{i=2,\ldots,n}\) are the solutions of

\[
\dot{x}_i = \frac{\partial f}{\partial x}(0,0)x_i + \frac{\partial f}{\partial u}(0,0)u_i, \\
x_i(0) = 0,
\]

then, for every \(i \in \{2, \ldots, n\}\),

\[x_i(T) = e_i.\]

Now let

\[b = \sum_{i=1}^{n} b_i e_i\]

be a point in \(\mathbb{R}^n\). Let \(v^1\) and \(v^2\), both in \(L^\infty((0, T); \mathbb{R}^m)\), be defined by the following

- If \(b_1 \geq 0\), then \(v^1 := v^1_+\) and \(v^2 := v^2_+\).
- If \(b_1 < 0\), then \(v^1 := v^1_-\) and \(v^2 := v^2_-\).
Then let \( u : (0, T) \rightarrow \mathbb{R}^m \) be defined by

\[
u(t) := |b_1|^{1/2}v^1(t) + |b_1|v_2(t) + \sum_{i=2}^{n} b_iu_i(t)\.
\]

Let \( x : [0, T] \rightarrow \mathbb{R}^n \) be the solution of

\[
\dot{x} = f(x, u(t)), \quad x(0) = 0.
\]

Then one has, as \( b \rightarrow 0 \),

\[
x(T) = b + o(b).
\]

Hence, using the Brouwer fixed-point theorem and standard estimates on ordinary differential equations, one gets the local controllability of \( \dot{x} = f(x, u) \) (around \( (0, 0) \in \mathbb{R}^n \times \mathbb{R}^m \)) in time \( T \), that is, for every \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that, for every \( (a, b) \in \mathbb{R}^n \times \mathbb{R}^n \) with \( |a| < \eta \) and \( |b| < \eta \), there exists a trajectory \((x, u) : [0, T] \rightarrow \mathbb{R}^n \times \mathbb{R}^m \) of the control system \( \dot{x} = f(x, u) \) such that

\[
x(0) = a, \quad x(T) = b, \\
|u(t)| \leq \varepsilon, \quad t \in (0, T).
\]
Bad and good news for $L = 2\pi$

- **Bad news:** The order 2 is not sufficient. One needs to go to the order 3
- **Good news:** the fact that the order is odd allows to get the local controllability in arbitrary small time. The reason: If one can move in the direction $\xi \in H^\perp$ one can move in the direction $-\xi$. Hence it suffices to argue by contradiction (assume that it is impossible to enter in $H^\perp$ in small time...)
With more controls: Global controllability result

\[ y_t + y_x + y_{xxx} + yy_x = u_0(t), \quad t \in [0, T], \quad x \in [0, L], \]
\[ y(t, 0) = u_1(t), \quad y(t, L) = u_2(t), \quad y_x(t, L) = u_3(t), \quad t \in [0, T]. \]

where, at time \( t \in [0, T] \),
- the control is \((u_0(t), u_1(t), u_2(t), u_3(t)) \in \mathbb{R}^4\),
- the state is \( y(t, \cdot) \in L^2(0, L) \).

**Theorem 3.6 (M. Chapouly (2008)).**

For every \( L > 0 \), and for every \( T > 0 \) the KdV control system (with four controls) is globally controllable in time \( T \).
Heuristics of the proof
Return method (Navier-Stokes, JMC (1996))
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Heuristics of the proof

Return method (Navier-Stokes, JMC (1996))

\[ \bar{y}(t) \]

\[ y \]

\[ B_1 \]

\[ B_2 \]

\[ t \]

\[ T \]
Heuristics of the proof

Return method (Navier-Stokes, JMC (1996))
Open problems

- Can one remove the control $u_0$ (i.e. $y_t + y_x + y_{xxx} + yy_x = 0$) and keep the global controllability result?
- With only one control ($y_x(t, L)$): Is there a minimal time for the local controllability for some $L \in \mathcal{N}$? (For a Schrödinger control system: JMC (2006).)
- “Rapid” stabilization. (For the linearized control system with only one control and $L \notin \mathcal{N}$: E. Cerpa and E. Crépeau (2008).)