Eigenvalue based techniques for the stability analysis and robust control of linear systems with time-delay

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Outline

- Motivating examples
- Basic properties of time-delay systems
- Computation of characteristic roots
- Fixed structure control design: an eigenvalue optimization approach
  - Stabilization via nonsmooth, nonconvex optimization
  - Computing and optimization robustness measures
- Case study: control of a heating system
- Concluding remarks
Motivating examples

- networks
  - biology (e.g. interactions between neurons)
  - car following models
  - time-based spacing of airplanes
  - distributed and cooperative control, sensor networks
  - congestion control in communication networks

- mechanical engineering
  - haptic interfaces
  - machine tool vibrations (cutting and milling machines)

- parallel computing (load balancing)

- population dynamics

- cell dynamics, virus dynamics

- laser physics (lasers with optical feedback)
Fluid flow model for a congested router in TCP/AQM controlled network

Hollot et al., IEEE TAC 2002

Model of collision-avoidance type:

\[ \dot{W}(t) = \frac{1}{R(t)} - \frac{1}{2} \frac{W(t)W(t-R(t))}{R(t-R(t))} p(t-R(t)) \]

\[ \dot{Q}(t) = \begin{cases} N(t) \frac{W(t)}{R(t)} - C & Q > 0 \\ \max \left( N(t) \frac{W(t)}{R(t)} - C, 0 \right), & Q = 0 \end{cases} \]

\[ R(t) = \frac{Q(t)}{C} + T_p \]

AQM is a feedback control problem: \( p = f(Q) \)

- \( W \): window-size
- \( Q \): queue length
- \( N \): number of TCP sessions
- \( R \): round-trip-time
- \( C \): link capacity
- \( p \): probability of packet mark
- \( T_p \): propagation delay
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Rotating milling machines

Model:
\[ \dot{x}(t) = F(x(t)) + B(\omega t) (x(t) - x(t - \tau(t))) \]

- Successive passages of teeth ⇒ delay
- Rotation of each tooth ⇒ periodic coefficients
- Delay inversely proportional to speed

Goal: increasing efficiency while avoiding undesired oscillations (chatter)
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Delays appear as intrinsic components of the system, or in approximations of (mostly PDE) models describing propagation and wave phenomena.
Heating system

Linear system of dimension 6, 5 delays
Model

\[
\begin{align*}
T_h \dot{x}_h(t) &= -x_h(t - \eta_h) + K_b x_a(t - \tau_b) + K_u x_{h,\text{set}}(t - \tau_u) \\
T_a \dot{x}_a(t) &= -x_a(t) + x_c(t - \tau_e) + K_a \left(x_h(t) - \frac{1+q}{2} x_a(t) - \frac{1-q}{2} x_c(t - \tau_e)\right) \\
T_d \dot{x}_d(t) &= -x_d(t) + K_d x_a(t - \tau_d) \\
T_c \dot{x}_c(t) &= -x_c(t - \eta_c) + K_c x_d(t - \tau_c) \\
\dot{x}_e(t) &= x_{c,\text{set}}(t) - x_c(t)
\end{align*}
\]

Control law (PI+ state feedback)

\[
x_{h,\text{set}} = K \begin{bmatrix} x_h & x_a & x_d & x_c & x_e \end{bmatrix}^T
\]
Representation as a functional differential equation

\[ C([-\tau, 0], \mathbb{R}^n) \]: Banach space of continuous function over \([-\tau, 0]\), equipped with the maximum norm, \( \| \cdot \|_s \)

functional \( f : C([-\tau, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n \)

Functional Differential Equation

\[
\begin{align*}
\dot{x}(t) &= f(x_t), \\
x(t) &\in \mathbb{R}^n, \quad x_t \equiv x(t + \theta), \quad \theta \in [-\tau, 0]
\end{align*}
\]

Linear Functional Differential Equations

\[
\dot{x}(t) = \int_{-\tau}^{0} d\theta [F(\theta)]x(t + \theta)
\]

F: bounded variation in \([-\tau, 0]\)

F(0)=0

⇒ unifying theory available

e.g., discrete delays \( \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t - \tau) \)
The initial value problem

Ordinary differential equation

\[
\frac{dx}{dt}(t) = f(x(t))
\]

linear

\[
\frac{dx}{dt}(t) = Ax(t)
\]

Delay differential equation

\[
\frac{dx}{dt}(t) = f(x(t), x(t - \tau))
\]

linear

\[
\frac{dx}{dt}(t) = A_0 x(t) + A_1 x(t - \tau)
\]

initial data required = function segment → infinite-dimensional system
Dynamics become rich when introducing a delay

**Analysis:** complex behavior

*scalar* examples

\[ \dot{x}(t) = -x(t - \pi/2) \quad \text{oscillatory solutions} \quad \dot{x}(t) = \sin t \]

\[ \dot{x}(t) = -20x(t) + 40 \frac{x(t - 1)}{1 + x(t - 1)^{10}} \quad \text{chaotic attractor} \]

**Controller synthesis**

any control design problem involving the determininination of a finite number of controller parameters is a low-order controller design problem

→ *inherent limitations*
→ control design almost exclusively ends up in an optimization problem
Reformulation in a standard, first order form

\[ \frac{dx}{dt}(t) = A_0 x(t) + A_1 x(t - \tau), \quad x(t) \in \mathbb{R}^n \]

\[ z(t) \equiv x(t + \theta), \quad \theta \in [-\tau, 0] \]

\[ \frac{dz}{dt}(t) = \mathcal{A} z(t), \quad z(t) \in X := C([-\tau, 0], \mathbb{R}^n) \]

where

\[ \mathcal{D}(\mathcal{A}) = \{ \phi \in X : \phi' \in X, \quad \phi'(0) = A_0 \phi(0) + A_1 \phi(-\tau) \} \]

\[ \mathcal{A} \phi = \phi' \]

\[ \rightarrow \text{a time-delay system is a distributed parameter system with a special structure: distribution in time} \]

\[ \rightarrow \text{“ambiguity”: infinite-dimensional system, but trajectories reside within a finite-dimensional space} \]
Ambiguity in the frequency domain

Linear(ized) time-delay systems: growth of solutions determined by spectrum

\[ \frac{d}{dt} z(t) = Az(t) \quad \Leftrightarrow \quad \dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) \]

\[ (\lambda I - A) u = 0 \quad u \in C([-\tau, 0], \mathbb{C}^n) \quad \Leftrightarrow \quad (\lambda I - A_0 - A_1 e^{-\lambda \tau}) v = 0 \quad v \in \mathbb{C}^n \]

infinite-dimensional
linear eigenvalue problem

\Leftrightarrow

finite-dimensional
nonlinear eigenvalue problem

Important element in developing numerical schemes:
exploiting two viewpoints
Example: computing characteristic roots via a two-step approach

\[(\lambda I - A)u = 0 \iff (\lambda I - A_0 - A_1 e^{-\lambda \tau})v = 0\] \hspace{1cm} (2)

1. discretize linear-infinite-dimensional operator; compute eigenvalues of the matrix
2. correct the individual characteristic root approximations using the nonlinear equation (2)
Large-scale problems: Krylov methods directly based on the infinite-dimensional representation

\[
(\lambda I - A_0 - A_1 e^{-\lambda \tau}) v = 0 \\
\Downarrow \\
(\lambda I - A) \phi = 0
\]

\(v\): vector

\(\phi\): function belonging to \(X := C([-\tau, 0], \mathbb{R}^n)\)

\[
\begin{align*}
\mathcal{D}(A) &= \{ \phi \in X \mid \phi' \in X, \phi'(0) = A_0 \phi(0) + A_1 \phi(-\tau) \} \\
A \phi &= \phi'
\end{align*}
\]

\(A\) derivative operator \(\Rightarrow A^{-1}\) integral operator

\[
\begin{align*}
\mathcal{D}(A^{-1}) &= X \\
A^{-1} \phi &= \int_0^t \phi(s) ds + C(\phi)
\end{align*}
\]

where

\[
C(\phi) = (A_0 + A_1)^{-1} \left( \phi(0) - A_1 \int_0^{\tau} \phi(s) ds \right)
\]
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◆ Motivating examples
◆ Basic properties of time-delay systems
◆ **Fixed structure control design: an optimization approach**
  ↗ Stabilization via nonsmooth, nonconvex optimization
  ↗ Computing and optimization robustness measures
◆ Case studies
  ↗ Control of a heating system
  ↗ Beneficial use of delays: prediction based feedback
Fixed structure control design

1. (infinite-dimensional) time-delay system

\[ \dot{x}(t) = \sum A_i x(t - \tau_i) + \sum B_i u(t - r_i) \]
\[ y_i(t) = \sum C_i x(t - s_i) \]

2. any type of controller characterized by finite number of parameters, \( p = (p_1, \ldots, p_m) \)

**static**

\[ u(t) = K y(t) \Rightarrow p = K \]

**dynamic**

\[ \begin{cases} \dot{z}(t) = F z(t) + G y(t) \Rightarrow p = (F, G, H) \\ u(t) = H z(t) \end{cases} \]

\[ u(t) = \sum K_i y(t - r_i) \Rightarrow p = (K_1, K_2, \ldots) / p = (K_1, r_1, K_2, r_2, \ldots) \]

⇒ closed loop system of the form

\[ \dot{z}(t) = \sum_{i=1}^{m} \tilde{A}_i(p) \ z(t - \tau_i) \]

⇒ control design = parameter tuning

= optimization of design specifications over the parameters
Fixed structure control design

1. (infinite-dimensional) time-delay system
   \[ \dot{x}(t) = \sum A_i x(t - \tau_i) + \sum B_i u(t - r_i) \]
   \[ y_i(t) = \sum C_i x(t - s_i) \]

2. any type of controller characterized by finite number of parameters, \( p=(p_1, \ldots, p_m) \)

\[ \Rightarrow \] closed loop system of the form
   \[ \dot{z}(t) = \sum_{i=1}^{m} \tilde{A}_i(p) \ z(t - \tau_i) \]

\[ \Rightarrow \] control design = parameter tuning
   = optimization of design specifications over the parameters

Motivation

- in applications the structure of the controller is mostly fixed or restricted
- a low order controller often perform well compared to full order controllers (a full order controller is infinite-dimensional)
- easy to implement
Objective function

Stabilization / response time

spectral abscissa function:

\[ c(p) = \max_{\lambda \in \mathbb{C}} \left\{ \Re(\lambda) : \text{det} \left( \lambda I - \sum_{i=0}^{m} A_i(p)e^{-\lambda \tau_i} \right) = 0 \right\} \]

characterizes the exponential decay of solutions. The system is stabilizable if and only if \( \min_p c(p) < 0 \)
neutral equation: \( \max(c(p), \bar{C}_D(p)) \)

\[
\dot{x}(t) = -x(t) + \frac{3}{4} x(t-1) + \frac{3}{4} \dot{x}(t-1) - \frac{1}{2} \ddot{x}(t-2)
\]
neutral equation: \( \max(c(p), \tilde{C}_D(p)) \)

\[
\dot{x}(t) = -x(t) + \frac{3}{4} x(t - 1)) + \frac{3}{4} \dot{x}(t - 0.99) - \frac{1}{2} \dot{x}(t - 2)
\]
neutral equation: \( \max(c(p), \tilde{C}_D(p)) \)

\[
\dot{x}(t) = -x(t) + \frac{3}{4} x(t - 1)) + \frac{3}{4} \dot{x}(t - (1-)) - \frac{1}{2} \dot{x}(t - 2)
\]
Robustness and performance

stable system:

\[ \dot{x}(t) = \sum A_i(p)x(t - \tau_i) + B(p)u(t) \]
\[ y(t) = C(p)x(t) + D(p)u(t) \]

transfer function:

\[ G(j\omega; \ p) := C(p) \left( j\omega I - \sum_{i=0}^{m} A_i(p)e^{-j\omega \tau_i} \right)^{-1} B(p) + D(p) \]

\( \mathcal{H}_\infty \) criterion

\[ \beta(p) := \|G(j\omega; \ p)\|_{\mathcal{H}_\infty} \]
\[ = \sup_{\omega \geq 0} \sigma_1(G(j\omega; \ p)) \]
$H_2$ criterion

$\dot{x}(t) = \sum A_i x(t - \tau_i) + Bu(t), \quad y(t) = Cx(t)$

$G(j\omega) := C \left( j\omega I - \sum A_i e^{-j\omega \tau_i} \right)^{-1} B$

$||G(j\omega)||_{H_2} = \left( \int_0^\infty \text{Tr} \ h(t)^* h(t) \ dt \right)^{1/2}$

$= \left( \frac{1}{2\pi} \int_0^\infty \text{Tr} \ G(j\omega)^* G(j\omega) d\omega \right)^{1/2}$

$h(t)$: impulse response

time domain

frequency domain
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◆ Basic properties of time-delay systems
◆ Fixed structure control design: an optimization approach
  ➣ Objective function
  ➣ Stabilization via nonsmooth, nonconvex optimization
  ➣ Computing and optimization robustness measures
◆ Case studies: control of a heating system
◆ Concluding remarks
Stabilization via nonsmooth, nonconvex optimization

Properties of the spectral abscissa function

\[ c(p) = \max_{\lambda \in \mathbb{C}} \left\{ \Re(\lambda) : \det \left( \lambda I - A_0 - \sum_{i=1}^{m} A_i(p) e^{-\lambda \tau_i} \right) = 0 \right\} \]

- not everywhere differentiable

- not locally Lipschitz continuous
• but ... smooth almost everywhere
Generalization of the steepest descent method
takes steps along the **nonsmooth steepest descent direction**:

\[-\arg \min_{z \in \delta_c \phi(p)} \|z\|, \quad \partial_c \phi(p) = \text{conv} \left\{ \lim_{q \to p} \tilde{\nabla} c(q) \right\}, \]

> generalized gradient (Clarke subdifferential) at \( p \)
The gradient sampling algorithm (Burke et al, SIOPT 2005)

- approximates the nonsmooth steepest descent direction by randomly sampling gradients in a neighborhood of the current iterate
The gradient sampling algorithm (Burke et al, SIOPT 2005)

- approximates the nonsmooth steepest descent direction by randomly sampling gradients in a neighborhood of the current iterate
- leads to a monotone decrease of the objective function towards a Clarke stationary point: $\tilde{0} \in \partial_c \phi(p)$

The algorithm relies on routines to compute the objective function and its gradient, whenever it exists.

- objective function: via computation of characteristic roots
- gradient: analytically or numerically (finite differences)

$$\frac{\partial \lambda(p)}{\partial p} = \frac{u^* \left( \sum \frac{\partial A_i(p)}{\partial p_i} e^{-\lambda \tau_i} \right) v}{u^* (I + \sum A_i(p) \tau_i e^{-\lambda \tau_i}) v}$$

- acceleration by BFGS
Coupled PDE-DDE model for a semiconductor laser

spatial discretization: DDE with dimension $n=123$
Coupled PDE-DDE model for a semiconductor laser

spatial discretization: DDE with dimension $n=123$

Invariant characteristic root at zero: due to symmetry
Computation of $\mathcal{H}_\infty$ norms

$$G(j\omega) := C \left( j\omega I - \sum_{i=1}^{m} A_i e^{-j\omega \tau_i} \right)^{-1} B + D e^{-j\omega \tau_0}$$

Principle criss-cross search

![Graph showing $\sigma_i(G(j\omega))$ vs $\omega$]
Main property

For $\omega \geq 0$, the matrix $G(j\omega)$ has a singular value equal to $\xi$ if and only if $\lambda = j\omega$ is an eigenvalue of the infinite-dimensional linear operator $\mathcal{L}_\xi$

\[\sigma_i(G(j\omega))\]

$\xi$

$j\omega_i$: eigenvalues of $\mathcal{L}_\xi$

$\mathcal{D}(\mathcal{L}_\xi) = \left\{ \phi \in C^1([-\tau_m, \tau_m]), \quad \phi'(0) = M_0(\xi)\phi(0) + \sum_{i=1}^m (M_i \phi(-\tau_i) + M_{-i} \phi(\tau_i)) + N_1(\xi)\phi(-\tau_0) + N_{-1}(\xi)\phi(\tau_0) \right\}$

$\mathcal{L}_\xi \phi = \phi'$, $\phi \in \mathcal{D}(\mathcal{L}_\xi)$.

Note: $\lambda \in \sigma(\mathcal{L}_\xi) \iff \det \left( \lambda I - M_0 - \sum_{i=1}^m M_i e^{-\lambda \tau_i} - \sum_{i=0}^m M_{-i} e^{\lambda \tau_i} - N_1 e^{-\lambda \tau_0} - N_{-1} e^{\lambda \tau_0} \right) = 0$
Predictor – corrector algorithm

$\mathcal{L}_\xi$ : infinite-dimensional operator

$\Rightarrow$ 2-step approach

1. criss-cross search using a finite-dimensional approximation (matrix) of $\mathcal{L}_\xi$

2. Newton like correction to peak value
Predictor – corrector algorithm

\( \mathcal{L}_\xi \): infinite-dimensional operator

\( \Rightarrow \) 2-step approach

1. criss-cross search using a finite-dimensional approximation (matrix) of \( \mathcal{L}_\xi \)

2. Newton like correction to peak value

Exploitation of duality eigenvalue problem of \( \mathcal{L}_\xi \)
- infinite-dimensional linear
- finite-dimensional nonlinear
Minimization of $\mathcal{H}_\infty$ norms

The function has the same properties as the spectral abscissa function, accelerated by BFGS. A stabilizing starting value can be generated by minimizing spectra abscissa.

$$G(j\omega; p) := C(p) \left( j\omega I - \sum A_i(p) e^{-j\omega \tau_i} \right)^{-1} B(p) + D(p)$$

- The function
  $$p \mapsto \|G(j\omega; p)\|_{\mathcal{H}_\infty}$$

  has the same properties as the spectral abscissa function
  
  → gradient sampling algorithm, accelerated by BFGS

- A stabilizing starting value can be generated by minimizing spectra abscissa
Computing of $\mathcal{H}_2$ norms

Finite-dimensional system

\[ G(j\omega) = C (j\omega I - A)^{-1} B \]

\[
\|G\|_{\mathcal{H}_2}^2 = \text{Tr} \left( B^T U B \right) = \text{Tr} \left( C V C^T \right)
\]

where

\[-C^T C = UA + A^T U \quad \text{(primal) Lyapunov equation}\]

and

\[-BB^T = VA^T + AV \quad \text{(dual) Lyapunov equation}\]

\[
\gamma(A, B, C) = \|G(j\omega; A, B, C)\|_{\mathcal{H}_2} \Rightarrow
\begin{cases}
\frac{\partial \gamma^2}{\partial A} = 2UV \\
\frac{\partial \gamma^2}{\partial B} = 2UB \\
\frac{\partial \gamma^2}{\partial C} = 2CV
\end{cases}
\]
Generalization to time-delay systems of retarded type

\[ G(j\omega) = C \left( j\omega I - A_0 - \sum_{k=1}^{m} A_k e^{-j\omega \tau_k} \right)^{-1} B \]

Approach 1: exploit nonlinearity of the characteristic matrix / representation as a functional differential equation
Generalization to time-delay systems of retarded type

\[ G(j\omega) = C \left( j\omega I - A_0 - \sum_{k=1}^{m} A_k e^{-j\omega \tau_k} \right)^{-1} B \]

\[ \|G\|^2_{\mathcal{H}_2} = \text{Tr} \left( B^T U(0) B \right) \]
\[ = \text{Tr} \left( C V(0) C^T \right) \]

where \( U \) and \( V \) are Lyapunov matrices, satisfying

\[
\begin{align*}
U'(t) &= U(t)A_0 + \sum_{k=1}^{m} U(t-\tau_k)A_k, \quad t \in [0, \tau_{\text{max}}] \\
U(-t) &= U^T(t) \\
-C^T C &= U(0)A_0 + A_0^T U(0) + \sum_{k=1}^{m} \left( U^T(\tau_k)A_k + A_k^T U(\tau_k) \right)
\end{align*}
\]

and

\[
\begin{align*}
V'(t) &= V(t)A_0^T + \sum_{k=1}^{m} V(t-\tau_k)A_k^T, \quad t \in [0, \tau_{\text{max}}] \\
V(-t) &= V^T(t) \\
-BB^T &= V(0)A_0^T + A_0 V(0) + \sum_{k=1}^{m} \left( V^T(\tau_k)A_k^T + A_k V(\tau_k) \right)
\end{align*}
\]

**Lyapunov equation → boundary value problem**

- explicit solution for commensurate delays
- general case: computation via discretization of boundary value problem based on spectral collocation
Computation of $\mathcal{H}_2$ norms

Finite-dimensional system

$$G(j\omega) = C (j\omega I - A)^{-1} B$$

$$\|G\|_{\mathcal{H}_2}^2 = \text{Tr} \left( B^T U B \right) = \text{Tr} \left( C V C^T \right)$$

where

$$-C^T C = U A + A^T U$$ \hspace{0.5cm} \text{(primal) Lyapunov equation}$$

and

$$-B B^T = V A^T + AV$$ \hspace{0.5cm} \text{(dual) Lyapunov equation}$$

$$\gamma(A, B, C) = \|G(j\omega; A, B, C)\|_{\mathcal{H}_2} \Rightarrow \begin{cases} \frac{\partial \gamma^2}{\partial A} = 2UV \\ \frac{\partial \gamma^2}{\partial B} = 2UB \\ \frac{\partial \gamma^2}{\partial C} = 2CV \end{cases}$$
Approach 2: exploit representation as a linear infinite-dimensional system

\[ G(j\omega) = C \left( j\omega I - A_0 - \sum_{k=1}^{m} A_k e^{-j\omega \tau_k} \right)^{-1} B \]

\[ = C(\lambda I - A)^{-1} B \rightarrow \text{Padé-via-Krylov model order reduction} \]

\[ \Rightarrow \|G(\lambda)\|_{H_2} - \|G_k(\lambda)\|_{H_2} = \mathcal{O} \left( k^{-3} \right) \]

complexity:
\[ \mathcal{O}(kn^3) + \mathcal{O}(k^3n) : \text{projection} \]
\[ \mathcal{O}(k^3) : H_2 \text{ norm reduced system} \]
Optimization of $\mathcal{H}_2$ norms

\[ G(j\omega) = C(p) \left( j\omega I - A_0(p) - \sum_{k=1}^{m} A_k(p)e^{-j\omega \tau_k} \right)^{-1} B(p) \]

• in contrast to the $\mathcal{H}_\infty$ norm, the $\mathcal{H}_2$ norm of $G$ smoothly depends on $p$, provided that the system matrices do so

• expressions for derivatives available

→ embedding in a derivative based optimization framework
→ second order methods applicable

<table>
<thead>
<tr>
<th>spectral abscissa, $\mathcal{H}_\infty$ norm</th>
<th>nonsmooth function of parameters</th>
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<tr>
<td>$\mathcal{H}_2$ norm</td>
<td>smooth function of parameters</td>
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</table>
Optimization of $\mathcal{H}_2$ norms

\[
G(j\omega) = C(p) \left( j\omega I - A_0(p) - \sum_{k=1}^{m} A_k(p)e^{-j\omega\tau_k} \right)^{-1} B(p)
\]

- in contrast to the $\mathcal{H}_\infty$ norm, the $\mathcal{H}_2$ norm of $G$ *smoothly* depends on $p$, provided that the system matrices do so
- expressions for derivatives available

→ embedding in a derivative based optimization framework
→ second order methods applicable

| spectral abscissa, $\mathcal{H}_\infty$ norm | nonsmooth function of parameters |
| smoothed spectral abscissa | (SIOPT 2009) |
| $\mathcal{H}_2$ norm | smooth function of parameters |
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◆ Motivating examples
◆ Basic properties of time-delay systems
◆ Fixed structure control design: an eigenvalue optimization approach
  ↪ Objective function
  ↪ Stabilization via nonsmooth, nonconvex optimization
  ↪ Computing and optimization robustness measures
◆ Case studies: control of a heating system
◆ Concluding remarks
Case study: control of a heating system

Model Vyhlídal, et al. (2009)

System
linear system, dimension 10, 7 delays

Extended state vector
\[ \bar{x}(t) = [x(t)^T \ I(t)]^T \]

\[ I(t) = \int_0^t (y_{SET}(\eta) - y(\eta))d\eta \]

\[ y, \ y_{SET} \]
- controlled variable and its setpoint

Controller
\[ u_c(t) = -K^T \bar{x}(t) \]

\[ u_c \] - control input

→ 11 free parameters
Objective of the control

• acceleration of the set-point response
• achieving a proper damping of the step and disturbance response

Approach

• minimizing the spectral abscissa
• subject to: pole location constraints

\[
\dot{x}(t) = \sum_{i=1}^{m} A_i(p) \ x(t - \tau_i)
\]

assigning a real pole \( c \):

\[
\det \left( cI - \sum A_i(p)e^{-c\tau_i} \right) = 0
\]

assigning a pair of complex conjugate poles \( c \pm dj \):

\[
\det \left( (c \pm dj)I - \sum A_i(p)e^{-(c \pm dj)\tau_i} \right) = 0
\]

Matrices \( A_i \) linear in \( p \)

→ polynomial constraints on parameters \( p \)

In addition: 1 control input

→ linear constraints, that can be eliminated
Stability optimization

spectrum of the open loop system

result of minimizing the spectral abscissa
Assigned poles:

\[ \lambda_1 = -0.025 \]
\[ \lambda_2 = -0.035 \]
\[ \lambda_{3,4} = -0.03 \pm 0.03i \]

Evolution of the objective function, and the gain values:
Set-point and disturbance responses
Concluding remarks

Introduction time-delay systems

Direct optimization approach for solving controller synthesis

• very suitable for designing reduced-order controllers
• generally applicable
• less restrictive than the existing time-domain methods

Combining different viewpoints

• time-domain versus frequency domain interpretations

• finite-dimensional nonlinear versus infinite-dimensional linear EVP: key towards new methods for generic nonlinear eigenvalue problems
Towards generic methods for nonlinear eigenvalue problems

“Linearization” of the eigenvalue problem:

\[(\lambda I - A_0 - A_1 e^{-\lambda \tau}) v = 0 \iff (\lambda I - A) z = 0\]

\[
\begin{align*}
\mathcal{D}(A) &= \{ \phi : \phi'(0) - A_0 \phi(0) - A_1 \phi(-\tau) = 0 \} \\
A \phi &= \phi', \quad \phi \in \mathcal{D}(A)
\end{align*}
\]

Generalization:

\[B(\lambda) v = 0 \iff (\lambda I - B) z = 0\]

\[
\begin{align*}
\mathcal{D}(B) &= \{ \phi : \left( B \left( \frac{d}{dt} \right) \phi \right)(0) = 0 \} \\
B \phi &= \phi', \quad \phi \in \mathcal{D}(B)
\end{align*}
\]