QUALITY MESHING OF MEDICAL GEOMETRIES WITH HARMONIC MAPS

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Two main research activities

Cardiovascular flow modeling: application to vascular bypasses (START-NHEMO Project)

PhD Students: Marie Willemet, Emilie Sauvage
Dr. Valérie Lacroix

Quality meshing for medical geometries
Today’s talk ...
What is Gmsh?

- OPEN SOURCE finite element grid generator with build-in CAD engine + post-processor.

http://www.geuz.org/gmsh
Motivation

Biomedical Engineering: geometries are triangulations

- Biomedical simulation requires high quality meshes
- Triangulations obtained from imaging techniques are of low quality
  - oversampled
  - non-delaunay triangulations
- Recover high quality surface mesh from low-quality inputs

Remeshing:

- Mesh adaptation
  Wang 2007, Bechet 2002, ...
- Parametrization of surface
  Floater 2005, Sheffer 2006, Marcum 1999, Levy 2004, ...
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Motivation (2)

CAD data is not suitable for FE analysis

- Geometric models of a landing gear
- CAD data issued form CATIA™
  - 852 surface patches
  - we were unable to build a suitable CFD mesh for that model
- Reparametrize through existing patches could be highly useful
  - 291 surface patches remaining
  - we were able to build a suitable CFD mesh for that model

Remeshing:

Cross-patch remeshing with based on cross-patch parametrization
Marcum 1999, Aftomosis 1999
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Remeshing:
Cross-patch remeshing based on cross-patch parametrization
Marcum 1999, Aftomosis 1999
Parametrizing a surface $S$ is **defining a map** $u(x)$

\[
x \in S \subset \mathbb{R}^3 \mapsto u(x) \in S^* \subset \mathbb{R}^2
\]  

(1)
Parametrizing a surface $S$ is defining a map $u(x)$

$$x \in S \subset \mathbb{R}^3 \mapsto u(x) \in S^* \subset \mathbb{R}^2$$ (1)

**Remesh with surface parametrization**

1. Compute the mapping $u(x)$
2. Remesh in the 2D space given the metric $M = x,u^T x,u$
3. Project the new mesh back to the 3D surface
Parametrizing a surface $S$ is defining a map $u(x)$

\[ x \in S \subset \mathbb{R}^3 \mapsto u(x) \in S^* \subset \mathbb{R}^2 \]  

Warning
Surfaces should have same topology
Outline

1. Computing maps
2. Max-cut partitioning
3. Automatic remeshing
4. Quality meshing
5. FE Biomedical simulations
1. Computing maps
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Minimize the Dirichlet energy:

\[ E_D(u) = \frac{1}{2} \int_S |\nabla u|^2 \, ds \]  \hspace{1cm} (2)

This quadratic minimization problem is equivalent to solving the two Laplace equations:

\[ \nabla^2 u = 0, \quad \nabla^2 v = 0, \quad \text{on} \ S \]  \hspace{1cm} (3)

with Dirichlet and Neumann boundary conditions

\[ u = u_D(x), \quad \text{on} \ \partial S_1, \quad \frac{\partial u}{\partial n} = 0, \quad \text{on} \ (\partial S - \partial S_1). \]  \hspace{1cm} (4)

The Radò-Kneser-Choquet theorem states that the harmonic mapping can be proven to be one-to-one, if surface \( S^* \) is convex.
Laplacian harmonic map with FE’s

- On the initial mesh, solve the Laplace equations with linear FE’s:
  \[
  u_h(x) = \sum_{i \in I} u_i \phi_i(x) + \sum_{i \in J} u_D(x_i) \phi_i(x) \tag{5}
  \]

- Solve a linear system
  \[
  \begin{pmatrix}
  \bar{A} & 0 \\
  0 & \bar{A}
  \end{pmatrix}
  \begin{pmatrix}
  \bar{U} \\
  \bar{V}
  \end{pmatrix} =
  \begin{pmatrix}
  0 \\
  0
  \end{pmatrix} \tag{6}
  \]

  with \( A_{kj} = \int_S \nabla \phi_k \cdot \nabla \phi_j \, ds \)

- Choose appropriate BC’s \( u_D(x) \), and \( v_D(x) \):
  \[
  u_D(x_i) = \cos(2\pi l_i/L), \quad v_D(x_i) = \sin(2\pi l_i/L), \tag{7}
  \]
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(7)
Convex combination map (Floater 1996)

Map the boundary nodes onto a well-known convex polygon (e.g. init circle) and place every interior vertex $u_i$ be the barycenter of its neighbors:

$$u_i = \sum_{k=1}^{d_i} \lambda_k u_j, \quad \sum_{k=1}^{d_i} \lambda_k = 1,$$

(8)

Solve a linear system

$$\begin{pmatrix} \bar{A} & \bar{0} \\ \bar{0} & \bar{A} \end{pmatrix} \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} = \begin{pmatrix} \bar{0} \\ \bar{0} \end{pmatrix}$$

(9)

with

$$A_{kj} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

(10)
Conformal maps preserve the angles:

Laplacian  Conformal
Minimize the conformal energy:

\[ E_{\text{LSCM}}(u) = \int_M \frac{1}{2} \left| \nabla u^\perp - \nabla v \right|^2 ds, \]  

(11)

where \( \perp \) denotes a counterclockwise 90° rotation in \( S \).
Minimize the conformal energy:
\[ E_{\text{LSCM}}(u) = \int_M \frac{1}{2} \left| \nabla u^\perp - \nabla v \right|^2 ds, \] (11)

where $\perp$ denotes a counterclockwise 90° rotation in $S$.

This minimization problem is equivalent to solving the following system:
\[
\begin{pmatrix}
\tilde{A} & \tilde{C} \\
\tilde{C}^T & \tilde{A}
\end{pmatrix}
\begin{pmatrix}
\bar{U} \\ \bar{V}
\end{pmatrix}
= \begin{pmatrix}
0 \\ 0
\end{pmatrix}
\]
(12)

where $\tilde{A}$ is a symmetric positive definite matrix $A_{kj} = \int_S \nabla \phi_k \cdot \nabla \phi_j \, ds$ and $\tilde{C}$ antisymmetric matrix $C_{kj} = \int_S n \cdot (\nabla \phi_k \times \nabla \phi_j) \, ds$. 
Minimize the conformal energy:

$$E_{LSCM}(u) = \int_M \frac{1}{2} \left| \nabla u^\perp - \nabla v \right|^2 ds, \quad (11)$$

where $\perp$ denotes a counterclockwise $90^\circ$ rotation in $S$.

This minimization problem is equivalent to solving the following system:

$$\begin{pmatrix} \bar{A} & \bar{C} \\ \bar{C}^T & \bar{A} \end{pmatrix} L_C \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} = \begin{pmatrix} \bar{0} \\ \bar{0} \end{pmatrix} \quad (12)$$

where $\bar{A}$ is a symmetric positive definite matrix $A_{kj} = \int_S \nabla \phi_k \cdot \nabla \phi_j \, ds$ and $\bar{C}$ antisymmetric matrix $C_{kj} = \int_S \mathbf{n} \cdot (\nabla \phi_k \times \nabla \phi_j) \, ds$.

How to assign proper boundary conditions?
Instead of solving $L_C u = 0$, make use of spectral theory:

The eigenvector $u^*$ (Fiedler vector) associated to the smallest eigenvalue $\lambda$, i.e. $L_C u^* = \lambda u^*$ is the solution to the constrained minimization problem:

$$u^* = \arg\min_{u, u^t e=0, u^t u=1} u^t L_C u$$

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Use efficient eigensolvers (Lanczos iterations, Choleski decomposition)
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Use efficient eigensolvers (Lanczos iterations, Choleski decomposition)

No need to pin down vertices (implicit constraints)
From continuous to discrete linear maps

Issues:

- undistinguishable coordinates
- triangle flipping
- triangle folding

Harmonic Laplacian mapping
From continuous to discrete linear maps

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Conformal mapping
Limitations for such a mapping?

Two topological conditions and one geometrical condition:
1. Surface of genus $G = 0$
2. Surface with at least one boundary $b$
3. Aspect ratio $\eta = H/D < 4$

How to compute those conditions?

The genus $G$ of the surface is computed from the Euler-Poincaré theory:

$$ G = \frac{b + \chi(S) - 2}{2} $$

where

- $b$ is the number of boundaries
- $\chi(S)$ is the Euler-Poincaré characteristic of the surface:
  $$ \chi(S) = \#V - \#E + \#F $$
Limitations for such a mapping?

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Partitioning the initial triangulation

Figure: Partitionning a lung with (left) a recursive multilevel method (Metis) and (right) a max-cut partitionner.
Idea of Alliez:

Define a split line that passes through the center of gravity and has the direction of the principal axis of inertia.

Figure: Partitioning a cylinder of aspect ratio $\eta = 25$. 
Max-cut partitioning based on Laplace map (Alliez et al 2002)
Max-cut partitioning based on multiscale Laplace map
Max-cut partitioning based on **multiscale** Laplace map
Max-cut partitioning based on multiscale Laplace map

Figure: Examples of the multiscale Laplace partitioning method: aorta(left) and human airways (right).
Figure: Remeshing of an arterial tree (a) with both the Laplacian harmonic map (b) and the conformal map (c).
1 Computing maps
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Automatic remeshing

a)

b)

c)

Remeshing algorithm.
- Initial triangulation \((G = 2, N_B = 0)\) that is cut into different mesh partitions of zero genus,
- Remesh the lines at the interfaces between partition
- Compute harmonic map for every partition and remesh the partition in the parametric space \((u(x)\) coordinates visible for one partition).
Automatic remeshing

Different mesh partitions of the skull in the parametric space (harmonic and conformal map)
Different mesh partitions of a hemi-pelvis in the parametric space (conformal map)
STL triangulations obtained from medical images (Left) that have been automatically remeshed with our automatic remeshing algorithm (Right).
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High quality meshing for the Laplacian harmonic map

Plot of the quality histogram with high mean quality \( \bar{\kappa} = 0.94 \):

\[
\kappa = \alpha \frac{\text{inscribed radius}}{\text{circumscribed radius}} = 4 \frac{\sin \hat{a} \sin \hat{b} \sin \hat{c}}{\sin \hat{a} + \sin \hat{b} + \sin \hat{c}}, \tag{14}
\]
Which mapping is best?

![Graphs showing comparison between Harmonic map, Conformal map, Convex map, and STL in terms of frequency versus aspect ratio.](image)
Which mapping is best?

Laplacian:

Convex

Initial mesh  
X-coord visible in map  
New mesh
High quality meshing

Comparisons with other techniques for surface remeshing:

- Harmonic map (Gmsh/meshadapt)
- Harmonic map (Gmsh/del2d)
- Local modifications (MeshLab)
- RIMLS (MeshLab)
- Local modifications (MAdLib)

CPU time < 100s for mesh of 1.e6 elements
High quality meshing

![Graph of frequency vs aspect ratio with lines for STL Mimics, Remeshed Mimics 1, Remeshed Mimics 2, and Remeshed Harmonic map.]
High quality meshing
Computing maps

Max-cut partitioning

Automatic remeshing

Quality meshing

FE Biomedical simulations
Blood flow in arterial anastomosis

Quality of the surface and volume meshes:

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Surface quality $\kappa_{min}$</th>
<th>Surface quality $\bar{\kappa}$</th>
<th>Volume quality $\gamma_{min}$</th>
<th>Volume quality $\bar{\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>STL</td>
<td>0.0033</td>
<td>0.821</td>
<td>0.0019</td>
<td>0.563</td>
</tr>
<tr>
<td>Remeshed</td>
<td>0.6400</td>
<td>0.949</td>
<td>0.2550</td>
<td>0.677</td>
</tr>
</tbody>
</table>
Blood flow in aortic arch

View a), b), c)

View b), c)
Blood flow simulation in an aortic arch. The left figure shows the WSS distribution and the right figure the WSS along the circumference at section $A - A'$ for different meshes for a constant inlet flow rate. The zero angle corresponds to the location $A'$. 
References

J.-F. Remacle, C. Geuzaine, G. Compère, E. Marchandise

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E. Marchandise, C. Carton de Wiart, W. Vos, C. Geuzaine, J-F. Remacle,
Conclusions

Presented:
- Automatic remeshing STL triangulations
- High quality surface meshes
- Appropriate for FE simulations (2D/3D)

Ongoing work:
- Spectral least square conformal map
- Hybrid method for conformal map (perform a few linear steps)
- Use conformal maps for generation of quad meshes (hex meshes)