Main Fields of Research

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Seminar Series of the Centre for Systems Engineering and Applied Mechanics (CESAME)
2.4. Transformée continue en ondelettes sur la sphère

Pour une certaine fonction

En remarquant que

Nommée également transformée de Fourier sur

we have chosen the parameters

fixings, are also better defined. Artifacts decrease in the bl

instance, that more isotropic features, such as Lena's righ

Cameraman results:

Fig. 2.3 – L'ondelette DOG pour

ψ

1

a

, and

2

= exp

− a

(2.67)

D = \{g_{\lambda}, \lambda \in \Lambda\}

Non-Local Processing

Compressed Sensing

Sparsity

Image Restorations

Compressed Imaging and Vision

Wavelets

Dictionaries and Matching Pursuit

\[ R^{m+1}f = R^m f - (g_{\lambda_{m+1}} : R^m f) g_{\lambda_{m+1}} \]

\[ \Phi \]

\[ x^* = \arg \min_u \| u \|_1 \text{ s.t. } y = \Phi u \]

Image Restorations

Non-Local Processing
avec 2.4. Transformée continue en ondelettes sur la sphère

condition homologue à l'annulation de la moyenne des ondes dont une représentation dilatée d'un facteur que [Van98]

En remarquant que

Pour l, m

standard deviation

we have chosen the parameters

fixed selectivity (26.32dB, Fig.10(c)). Isotropic feature

Fig. 8. Denoising of the instance, that more isotropic features, such as Lena's right ears, than that of the fixed selectivity method (27.72dB, Fig. 9(d)) than that of the fixed selectivity method (27.72dB, Fig.

t nostril or the tip of her nose, are better preserved in the ack area of the cameraman's coat.

Matching Pursuit

Extensions (rnd. proj. crypto):

Prof. M. Jalal Fadili

Salomeh Shariati
Dequantizing Compressed Sensing:
When Oversampling and Non-Gaussian Constraints Combine

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The Problem

- Compressed Sensing says:
  “Linearly sample a signal at a rate function of its intrinsic dimensionality”

- Sensor designer (& Inf. Th.) says:
  “Okay, but I need to quantize/digitize (ADC) my measurements!”

- Question:
  “Given quantized signal measurements, how to minimize quantization effects in the reconstruction?”

- Our answer:
  “Oversample and reconstruct with non-gaussian constraints, i.e. in $\ell_p$”
Signal Sparsity (1/2)

* Signals have structures, features, edges, ...

3-D data

Speech signal

biology

Video

astronomy
Signal Sparsity (2/2)

* Given $x \in \mathbb{R}^N$ (e.g. $N =$ number of pixels, samples, voxels ...)

* There is a sparsity basis (e.g. Wavelets, DCT, Fourier)

\[ \Psi = (\Psi_1, \ldots, \Psi_D) \in \mathbb{R}^{N \times D} \]

where $x$ has the representation

\[ x = \sum_{j=1}^{D} \alpha_j \Psi_j = \Psi \alpha \]

$\alpha \in \mathbb{R}^D$ is the coefficient vector

* True sparsity: $\alpha$ has $K$ non-zero elements

* Compressibility: $|\alpha(k)| \leq C k^{-p}$  
  (sparsely approximable)
Compressed Intro to “Compressed Sensing”  (1/3)

* Given a sparse (or compressible) signal  \( x \in \mathbb{R}^N \), i.e.
  \[
  x = \Psi \alpha, \quad \|\alpha\|_0 \leq K, \quad (K \ll N)
  \]
  for a sparsity basis \( \Psi \in \mathbb{R}^{N \times N} \) (assume \( \Psi = \text{Id} \))

* **Common sampling** [Shannon, Nyquist] + **Compression**: record/sample all the \( x_i \) and keep the largest \( \alpha_i \)

* **Compressed Sensing** [Candes, Tao, Romberg, Donoho, ... 2006]:
  * “forget” Dirac, forget Nyquist, ... and sample just linearly!
  * given \( M < N \) sensing vectors \( \varphi_i \in \mathbb{R}^N \),
    \[
    y_i = \langle \varphi_i, x \rangle, \quad y = \Phi x \in \mathbb{R}^M
    \]
  * for a measurement “rate”:
    \[
    M \propto \text{signal intrinsic dimension (i.e. sparsity } K)\]

\[
\begin{align*}
\Phi & = \begin{bmatrix} y_1 & \cdots & y_M \end{bmatrix} \\
\Phi & \in \mathbb{R}^{M \times N} \\
\end{align*}
\]
Compressed Intro to “Compressed Sensing” (2/3)

* Good matrices are RIP matrices of order $2K$:

$$
\exists \ c > 0, \ \delta \in (0, 1) \quad \text{Restricted Isometry Property}
$$

$$
c \sqrt{1 - \delta} \|v\|_2 \leq \|\Phi v\|_2 \leq c \sqrt{1 + \delta} \|v\|_2,
$$

for all $2K$ sparse signals $v$.

* Random constructions: RIP w.h.p. if

Gaussian $N(0, 1)$, Bernoulli $\pm 1$, Random Fourier/ONB ensemble, ....

$$
M = O(K \log N/K), \ c = \sqrt{M} \quad M = O(\mu(\Phi, \Psi)^2 K \log^4 N)
$$

* Ideal solver (NP hard): if $\Phi$ is RIP $2K$, $x = \Delta_0(y)$

$$
\Delta_0(y) = \arg \min_{v \in \mathbb{R}^N} \|v\|_0 \ \text{s.t.} \ y = \Phi v.
$$
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* Ideal solver (NP hard): if $\Phi$ is RIP $2K$, $x = \Delta_0(y)$

\[ \Delta_0(y) = \arg \min_{v \in \mathbb{R}^N} \|v\|_1 \text{ s.t. } y = \Phi v. \]  

(Basis Pursuit - BP)  

\[ \text{if } \delta < \sqrt{2} - 1 \quad [\text{Candes 08}] \]
Basis Pursuit ... intuitively

\[ x = x_{BP} \]

\[ \Phi u = y \]

\[ x_{BP} = \arg \min_u \|u\|_1 \text{ s.t. } y = \Phi u \]
Compressed Intro to "Compressed Sensing" (3/3)

* When noise happens, i.e. $y = \Phi x + n$, $n_i \sim \mathcal{N}(0, \sigma^2)$, 
  → use of Basis Pursuit DeNoise:

  \[
  \Delta_{1,2}(y, \epsilon) = \arg\min_{v \in \mathbb{R}^N} \|v\|_1 \text{ s.t. } \|y - \Phi v\|_2 \leq \epsilon, \quad \text{(BPDN)}
  \]

* Stability (wrt compressibility & noise): $\ell_2 - \ell_1$ instance optimality

  \[
  \|x - \Delta_{1,2}(y, \epsilon)\|_2 \leq A e_0(K) + B \epsilon/c,
  \]

  with $e_0(K) = \|x - x_K\|_1/\sqrt{K}$, $A$ and $B$ reasonable function of $\delta$.

* Noise power estimator: $\|n\|_2^2 \leq \epsilon^2 = \sigma^2 (M + \kappa 2\sqrt{2M})$

  w.h.p.
Quantization and CS: the former association

- Model: Uniform Quantization of Measurements (no saturation)

\[ y = Q_\alpha[\Phi x] \in \mathbb{R}^M, \]

with \((Q_\alpha[y])_i = \alpha[(y)_i/\alpha] + \alpha/2\).

- Reconstruction: since \(\|y - \Phi x\|_\infty \leq \frac{\alpha}{2}\)

- quantization \(\approx\) uniform noise: \(u_i = y_i - (\Phi x)_i \sim_{\text{iid}} \mathcal{U}([-\frac{\alpha}{2}, \frac{\alpha}{2}])\)

- BPDN with \(\epsilon^2_2(\alpha) = E[\|u\|_2^2] + \kappa \text{Var}^{\frac{1}{2}}[\|u\|_2^2]\)

\[ = M \frac{\alpha^2}{12} + \kappa M^{\frac{1}{2}} \frac{\alpha^2}{6\sqrt{5}} \]

- But, not adapted: if \(x^*\) solution of BPDN,

  - \(Q_\alpha[\Phi x^*] \neq y\), i.e. no Quantization Consistency (QC)
  - \(\ell_2\) constraint \(\approx\) Gaussian distribution (MAP - cond. log. lik.)
New class of solvers : BPDQ\(_p\)

- New optimization schemes (towards the constraint \(\|y - \Phi v\|_\infty\)):

\[
\Delta_{1,p}(y, \epsilon) = \arg \min_{v \in \mathbb{R}^N} \|v\|_1 \text{ s.t. } \|y - \Phi v\|_p \leq \epsilon, \quad (\text{BPDQ}\_p)
\]

for “Basis Pursuit DeQuantizer” of moment \(p \geq 1\)

BPDQ\(_p\) adapted to GGD noise (of shape \(p\)): \(n_i \sim \text{pdf} \propto \exp -|t/b|^p \) (\(b > 0\))

- For \(y = Q_\alpha[\Phi x] \in \mathbb{R}^M\), (assuming uniform distribution)

\[
\epsilon \leftarrow \epsilon_p(\alpha) = \frac{\alpha}{2(p+1)^{1/p}} \left( M + \kappa(p+1)\sqrt{M} \right)^{1/p}
\]

then \(\mathbb{P}[\|y - \Phi x\|_p \geq \epsilon_p] \leq e^{-2\kappa^2}\) (i.e. \(x\) is feasible whp)

and \(\epsilon_p(\alpha) \xrightarrow{p \to \infty} \frac{\alpha}{2} = QC\)!

- Implicitly: “Is \(p = \infty\) the best for quantization?” [Candes, Tao, 06]

\[\Rightarrow \text{ Study BPDQ}_p \text{ on } p \in [2, \infty]\]
RIP\textsubscript{\(p\)} and Approximation Error Bounds \((1/2)\)

* Are these solvers RIP consistent? Can we bound the approximation error between the sparse or compressible \(x\) and

\[
x_p^* = \Delta_{1,p}(y, \epsilon_p(\alpha))?
\]

* We need a new isometry: \(\Phi\) is RIP\textsubscript{\(p\)} of order \(K\) if

\[
\exists \mu_p > 0, \delta \in (0,1), \quad \mu_p \sqrt{1 - \delta} \|v\|_2 \leq \|\Phi v\|_p \leq \mu_p \sqrt{1 + \delta} \|v\|_2,
\]

for all \(K\) sparse signals \(v\).

* Good news: \(\Phi \in \mathbb{R}^{M \times N}, \Phi_{ij} \sim_{iid} \mathcal{N}(0,1)\), is RIP\textsubscript{\(p\)} (whp, for \(2 \leq p < \infty\),

\[
M = O((K \log N/K)^{p/2}) \quad \text{if} \quad M \propto M^{1/p} \sqrt{p + 1} (1 + O(p^{-2}))
\]

[\text{Lipschitz embedding of the} \ell_p \text{ norm: Ledoux, Talagrand}]

\(\Rightarrow\) Another reading: limited range of valid \(p\) for a given \(M\) (and \(K\))!
RIP_p and Approximation Error Bounds (2/2)

Therefore, for M big enough (\ell_2 - \ell_1 instance optimality)

\[
\|x - \Delta_{1,p}(y, \epsilon_p)\|_2 \leq A_p e_0(K) + B_p \frac{\epsilon_p(\alpha)}{\mu_p} \\
= A_p e_0(K) + B_p \frac{\epsilon_p(\alpha)}{\sqrt{p+1}} \left(1 + O(p^{-2})\right)
\]

Potentially smaller errors than for \( p = 2 \) (i.e. BPDN)! but only for \( M \) sufficiently high (oversampling principle) since:

\[
A_p = A_p(\delta) \geq 2, \\
B_p = B_p(\delta) \geq 4, \\
\delta \propto M^{-1/p}.
\]
Numerical Methods

- BPDQ\(_p\) solved by proximal optimization (Douglas - Rachford)
  \[
  \arg \min_{v \in \mathbb{R}^N} f(v) + g(v), \quad f(v) = \|v\|_1, \quad g(v) = \nu_{T^p(\epsilon)}(v)
  \]
  with \(\nu_A(v) = 0\) if \(v \in A\) and \(\infty\) else, \(T^p(\epsilon) = \{v : \|y - \Phi v\|_p \leq \epsilon\}\)

- But, you need a projector onto the fidelity constraint
  ... you need to project onto a \(\ell_p\) ball \(\{y' : \|y' - y\|_p \leq \epsilon\}\)

- for \(p \in \{1, 2, \infty\}\) easy! (resp. soft-thresholding, radial projection, minimum projection)

- else, well... use iterative methods (Augmented Lagrangian problem + Newton)

- Matlab BPDQ toolbox available ([http://wiki.epfl.ch/bpdq](http://wiki.epfl.ch/bpdq))
  with 1-D and 2-D reconstruction demonstrations
Reconstruction Results (1/3)

* $N=1024$, $K=16$, Gaussian $\Phi$
* 500 $K$-sparse (canonical basis)
* Non-zero components follow $\mathcal{N}(0, 1)$
* Quantiz. bin width $\alpha = \|\Phi x\|_\infty / 40$
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Histograms of $\alpha^{-1}(y - \Phi x_p^*)_i$

$M \rightarrow 2M$, $\alpha$ cst. $
\iff$
$\alpha \rightarrow \alpha/2$, $M$ cst.

$\ast$ $N=1024$, $K=16$, Gaussian $\Phi$
$\ast$ 500 $K$-sparse (canonical basis)
$\ast$ Non-zero components follow $\mathcal{N}(0, 1)$
$\ast$ Quantiz. bin width $\alpha = \|\Phi x\|_{\infty}/40$
Reconstruction Results (2/3)

A bit outside the theory...

* Synthetic Angiogram [Michael Lustig 07, SPARCO],
* Φ: Random Fourier Ensemble
* $N/M = 8$
* Decoder: $\Delta_{TV,p}(y, \epsilon_p)$
* Quantiz. bin width = 50 (i.e. 12 bins)

BPDN-TV
SNR: 8.96 dB

BPDQ_{10} -TV
SNR: 12.03 dB
Reconstruction Results

Optimal moment $p$ for angiogram for $N/M = 8$:

![Graph showing SNR against $p$]

Average SNR (solid) and SNR improvement (dashed) over BPDQ vs $p$. Tests made on 50 trials on the Random Fourier Ensemble with $N/M = 8$. 
Conclusions and perspectives

- Quantization distortion better handled with $\ell_p$ norm constraint for an optimal $p = p(M)$.

- How to find optimal $p$? Possible heuristic?

- How to integrate the quantization saturation? [J. Laska et al., 2009]

- Does a greedy dequantizer exist? [W. Dai et al, 2009]

- Gaussian noise before quantization? [A. Zymnis et al. 2009]
Links (Science 2.0.)

* Rice CS Resources page: http://www-dsp.rice.edu/cs

* Igor Carron’s “Nuit Blanche” blog: http://nuit-blanche.blogspot.com
  1 CS post/day!
References on Compressed Sensing:


Some references on Quantized CS:


Thank you!