Short-run and long-run marginal costs of joint products in linear programming

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1 Introduction

1.1 The standard microeconomic theory

When one considers an activity involving the production of a single product, microeconomic theory tells us that with adjusted capacity, short-run and long-run marginal costs are equal under certain conditions. To take a specific example, consider a construction project for a facility of which the size (i.e., the production capacity) needs to be determined. We assume that this size can be represented by a continuous variable. We also assume that the annual production cost for a given size, i.e. the short-run annual cost, is a continuous function that can be differentiated with respect to the quantity of good produced over the year. If the marginal cost begins to increase at a certain level of production, the curves representing the annual cost for facilities of various sizes generally take the shape of those drawn by the dashed lines on the first graph of Figure 1. The curve representing the long-run annual cost is then the envelope curve (shown by a solid line) of the short-run cost curves. The second graph of Figure 1 reveals, in the case of a rising marginal cost over the long run (long dashed lines), the usual shape of average cost curves over the short run (dashed lines) and the long run (solid line), as well as curves representing short-run and long-run marginal cost (it should be noted that marginal cost can decrease over the long-run in the presence of economies of scale). Under the usual conditions of continuity and differentiability, these properties are well known (e.g. see Boîteux 1960).

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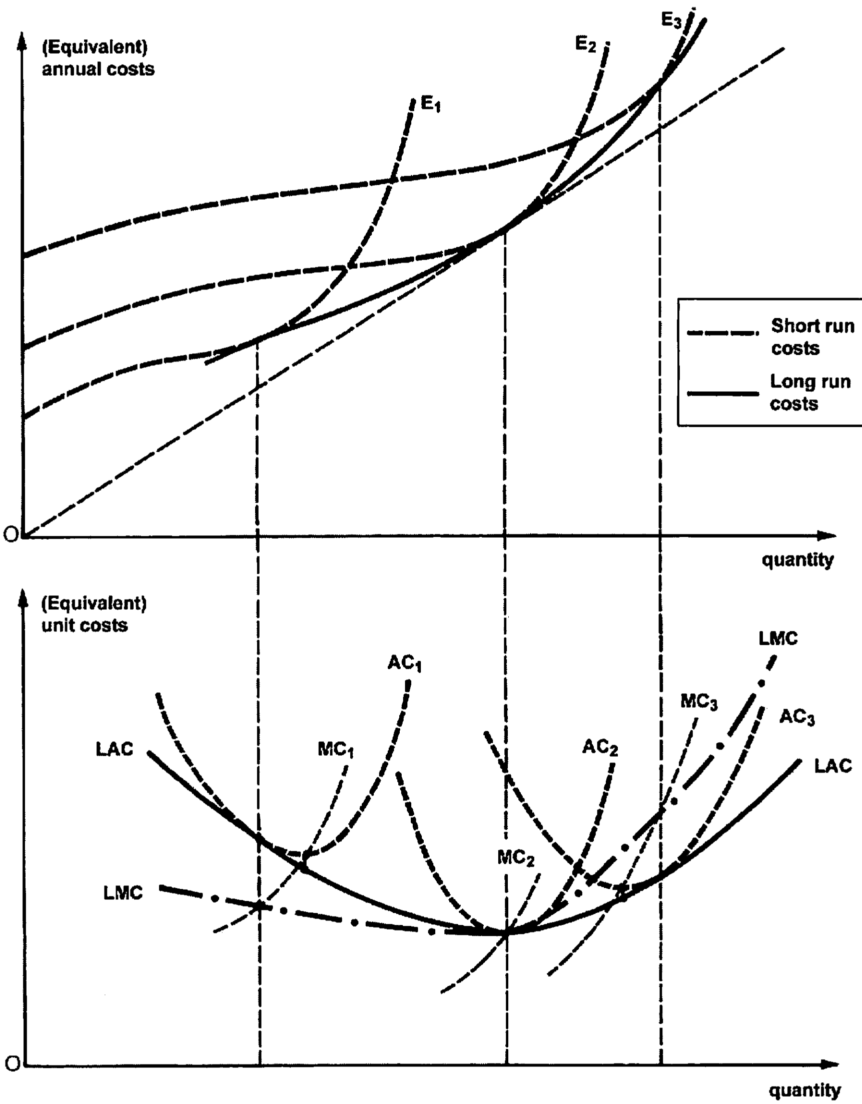

Figure 1 : *Equality of short-run and long-run marginal costs in standard micro-economic theory*

LAC: long-run average cost; LMC: long-run marginal cost; AC_k: short-run average cost with a production facility of size E_k; MC_k: short-run marginal cost with a production facility of size E_k.

1.2 Purpose of this paper

In this article, we analyze the case of the production of joint products using various types of interdependent equipment when the production system is modeled via a linear program. Our objective is to establish the relationship between short-run marginal costs and long-run marginal costs in this case.

A first example of such a production system is given by the petroleum refining industry: the main variables represent the quantities of crude oil to be processed and the various flows of intermediate products that characterize the operation of the production units (atmospheric and vacuum distillation, reforming, cracking, etc.) and the composition of the finished products (gasoline, automotive diesel, heating oil, heavy fuel oil, etc.). The main constraints are material-balance equations (which are the most common), product-quality specification equations (sulphur content, gasoline octane number, etc.) and demand equations for finished products (quantities produced, increased in some cases by imported quantities and decreased by exported quantities, greater than or equal to demand). In a short-term management model, there are also capacity equations limiting the feedstock to be processed in each of the units. The economic function only includes operating costs (decreased where applicable by revenues generated by the sale of certain products). In a long-term investment model, variables represent the capacities of the units to be built. If we are dealing with a dynamic model extending over several periods, (linear) investment costs must be associated with these capacity variables, and the economic function is then a total cost discounted over a long period. If we are dealing with a "static" long-term model, the model is representative of a given time horizon. An equivalent annual investment cost is then associated with each capacity variable (initial investment divided by the sum of the discount factors). The economic function to be minimized is then an equivalent annual cost, which results from the sum of the operating cost and equivalent annual investment costs.

Yet another example is supplied by an electrical production system using various types of power plants (nuclear, coal, gas turbine, etc.). The main variables are the powers supplied by the different types of plant for the various hourly/seasonal periods (peak periods during winter weekdays, off-peak periods during summer weekends, etc.). The main constraints are the demand equations expressing the requirement to supply the power demanded for each hourly/seasonal period (and for each outcome if the demand is randomized). Anderson (1972) offers a general description of this type of model.

In practice, the models used for short-term management (particularly in the very short run), are often profit maximization programs, with the price of products being fixed and assumed to be given by the market. It is always possible, however, by taking certain precautions where appropriate, to make the transition from such a profit-maximization formulation to a cost-minimization formulation, subject to the constraint of satisfying a demand for finished products. It is this latter formulation that will be used in the remainder of this article.

We will therefore consider a system composed of a variety of equipment, called production units, producing several joint products. The interdependencies between the units are represented via a linear programming model. We distinguish between a short-term program, for which the capacity of the units is assumed to be fixed, and a long-term program for which a certain number of capacities are variables to be determined. Equivalent annual investment costs are then associated with these variables. For purposes of simplification, we will present the analysis by referring to a static long-term model representative of a given time horizon. Over both the short- and long-term, we will consider a cost-minimization problem under demand constraints.

1.3 Preliminary remarks on marginal costs

We will begin with some general remarks. Marginal production costs are equal to the dual variables associated with demand constraints, at least if there is no degeneracy. When one varies the demand for a given product, while keeping the demand for other products unchanged, the marginal cost of that product is stable as long as the optimal solution corresponds to the same basic solution. The curve representing the marginal cost as a function of the quantity produced is therefore a stair-step curve. This is true for both the short-run marginal cost and the long-run marginal cost.

Let's consider the optimal solution of a long-term problem, determined by taking into account a fixed demand b_i for each product i. This solution gives the optimal value for the capacities to be built for the various units. Now set these capacities at their optimal values. The short-term program thus obtained supplies the values for the short-run marginal costs and, in more general terms, allows us to trace the short-run marginal cost of a product as a function of the demand for that product. The short-run marginal cost curve for a product i intersects the long-run marginal cost curve for a demand equal to b_i, with the capacities consequently being adjusted. This is in line with standard microeconomic theory. The distinguishing feature of a production system represented by linear programming is related to the stair-step profile of the curves represented in Figure 2. In general, the long-run marginal cost is stable in the neighborhood of b_i while one observes a discontinuity of the short-run marginal cost.

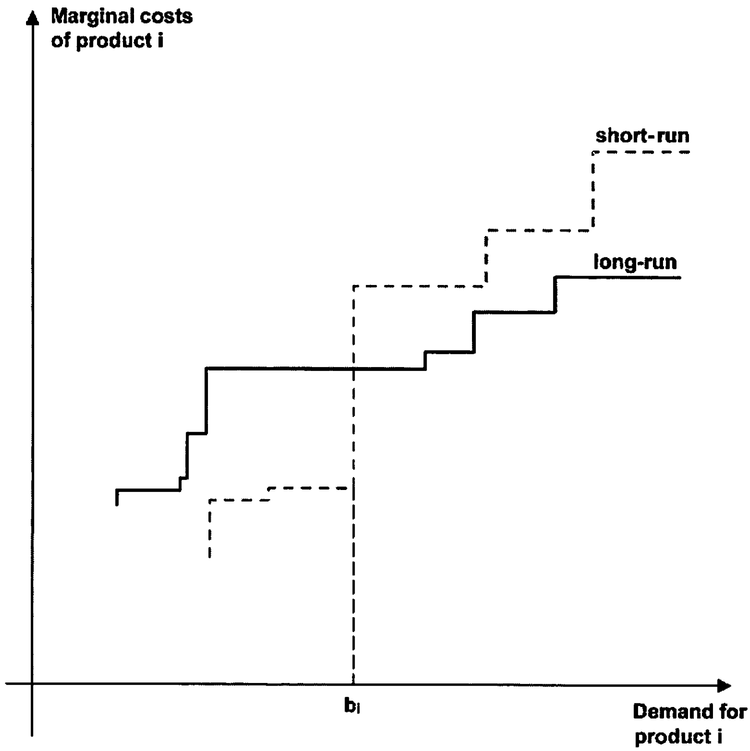

Figure 2 : *Short-run and long-run marginal costs of product* i *in linear programming, with capacity adjusted to demand* b_i

In fact, a long-term problem can be converted into a short-term problem by adding constraints setting the capacities of the units, requiring that the feedstock processed by a unit, or its production, be less than the capacity defined by the optimal solution of the long-term program. These constraints, or at least a certain number of them, are binding at the optimum of the short-term program, which is then degenerate. The objective-function value of the program is no longer differentiable in b_i. Nevertheless, it has a sub-differential and directional derivatives, which explains the discontinuity of the short-run marginal cost observed at b_i in figure 2.

For each finished product, the short-run marginal costs at the left side and right side of the anticipated demand for which the capacity is adjusted are then different and given by components of the dual variables vectors. The literature on degeneracy supplies the criteria for selecting these components in the presence of multiple dual solutions (Aucamp and Steinberg 1982, Gal 1986 and Greenberg 1986). Consequently, we must consider a left-hand marginal cost and a right-hand marginal cost for each finished product.

The objective of this article is to study the relationship between short-run marginal costs and long-run marginal costs. The analysis relies on the marginal values of capacities, equal to the dual variables associated with the capacity constraints introduced in the short-term model. In theory, these variables are negative: in the short run, an increase in available capacity should not generally result in an increase in the operating cost. In order to simplify the presentation of results, we assume throughout this article that the marginal values of capacities are all negative.

1.4 Breakdown of long-run marginal costs

We use the breakdown of long-run marginal costs in linear programming proposed by Pierru and Babusiaux (2004): the long-run marginal cost of a given finished product results from the sum of a marginal operating cost and a marginal equivalent investment cost. To obtain it, one simply breaks down the objective function into two elementary economic functions [1]. The marginal operating cost (respectively the marginal equivalent investment cost) is equal to the variation of the elementary function representing the operating cost (respectively the equivalent annual investment cost) for an infinitesimal increase in finished product demand. It has to be noted that the marginal equivalent investment cost is equal to the marginal capacity adjustment times the equivalent annual investment cost per unit of installed capacity.

We will adopt this breakdown with the comment that the equivalent annual investment cost taken into account can result from the sum of the investments in the various units whose capacities are to be optimized. In addition to the operating cost, we will therefore associate an elementary function with each capacity variable. The long-run marginal cost of a product can therefore be broken down into a marginal operating cost and as many marginal equivalent investment costs as there are capacity variables. To eliminate any ambiguity, we specify that for a given finished product the term "marginal operating cost" always refers in this article to a change in the operating cost determined in the long-term model.

The following section introduces the mathematical formulation of the model and the notations used. The results are then presented in the form of propositions. For reasons of clarity, in section 3, we first state and prove the propositions when the model only includes a single capacity variable. We then state and prove those pertaining to the general case in section 4. A numerical illustration is proposed in section 5 based on the simplified refining model developed by Pierru and Babusiaux in their article.

[1] With each elementary objective function these authors associate a vector composed of "elementary dual variables"; we will not review the method for breaking down long-run marginal costs here.

2 Notations and mathematical formulation of the model

To illustrate the problem under study, we will consider a static long-term linear-programming model and the corresponding short-term linear-programming model. The long-term model is used to define the optimal capacities to be built to meet a demand vector b and provides the long-run marginal cost of each finished product, as well as its breakdown into marginal operating cost and marginal equivalent investment costs. The short-term model determines the optimal production program for demand b, assuming that capacities are set at the values defined by the long-term model. This short-term model is used to calculate the marginal values of the capacities, as well as the short-run marginal costs (on the left-hand and right-hand of demand b_i) for each finished product i. Although the problem studied relates to a linear programming model, we will adopt a formulation that takes its inspiration from the lagrangian duality (which facilitates the formal application of the envelope theorem).

2.1 Long-term model

We consider a capacity variable and a feedstock variable for each equipment. A feedstock variable has to be smaller or equal to the corresponding capacity variable (plus a possibly pre-existing capacity). With this formulation, the operating cost (associated with the feedstock variable) can be distinguished from the equivalent annual investment cost (associated with the capacity variable) in the objective function.

We adopt the following notations, considering that the long-term problem includes m capacity variables:

b: vector formed from the demanded quantities of n finished products (the quantity of product i demanded is denoted b_i);

x: vector of size q formed from the variables which appear in both long-run and short-run primal problems;

x_f: subvector of x whose components are the m feedstock variables;

$k = (k_1, k_2, ..., k_m)$: vector formed from the m capacity variables in the long-term problem;

(x, k) thus represents the entire set of variables of the long-term problem;

$e = (e_1, e_2, ..., e_m)$: vector formed from the equivalent annual investment costs per unit of installed capacity, building k_p $(p \in \{1, 2, ..., m\})$ involves an equivalent annual investment cost equal to $e_p k_p$;

The long-term problem is written as follows:

$$Min \langle v, x \rangle + \langle e, k \rangle$$

$$s.t. \begin{cases} x \in X \\ Ax = b \\ x_f - k \leq c \end{cases}$$

Where: A is an $n \times q$ matrix, $Ax = b$ represents the set of demand constraints; X is a closed convex set defined as the intersection of linear constraints (equalities and inequalities), corresponding to material balance equations, product-quality specifications, requirements on the signs of variables ... c is a vector whose components are the pre-existing capacities of the m types of equipment considered; v is a vector of size q whose components are the unit costs associated with the variables in x; $\langle .,. \rangle$ denotes the inner product of two vectors.

The value of this long-term problem, considered as a function of the demand for finished products, is notated $V(b)$. We assume that the optimal solution is non-degenerate; the basic solution is thus unchanged in the neighborhood of b. $V(b)$ is therefore continuously differentiable. We will let $l_i(b)$ represent the long-run marginal cost of the product i: $l_i(b) = \frac{\partial V}{\partial b_i}(b)$. It is equal to the dual variable associated with the demand constraint for that product.

The value of k at the optimum of the long-term problem is denoted $\hat{k}(b)$, with $\hat{k}(b) = (\hat{k}_1(b), \hat{k}_2(b), ..., \hat{k}_m(b))$. A change in the demand for finished product i requires a marginal capacity adjustment equal to $\frac{\partial \hat{k}_p}{\partial b_i}$ in order for the p^{th} capacity to remain adjusted. This marginal capacity adjustment generates a marginal equivalent investment cost equal to $e_p \frac{\partial \hat{k}_p}{\partial b_i}$.

2.2 Short-term model

The short-term problem, derived from the long-term problem introduced in the previous subsection, is written as follows:

$$Min \langle v, x \rangle$$

$$s.t. \begin{cases} x \in X \\ Ax = b \\ x_f \leq c + \hat{k}(b) \end{cases}$$

We introduce two vectors of dual variables, associated with two sets of constraints in this short-term problem:

$y = (y_1, y_2, ..., y_n)$ vector formed from the dual variables associated with the demand constraints in the short-term problem, where y_i

($i \in \{1, 2, ..., n\}$) is the dual variable associated with the demand constraint for product i;

$u = (u_1, u_2, ..., u_m)$: vector formed from the dual variables associated with capacity constraints in the short-run, where u_p ($p \in \{1, 2, ..., m\}$) is the dual variable associated with the p^{th}-capacity constraint.

2.3 Construction of a "short-run cost function with continuously-adjusted capacity"

Let W be a function correctly defined in a neighborhood of b, representing the value of the short-term program when the capacity is always adjusted. W is equal to V (the long-term and short-term problems have the same optimal solution) less the sum of the equivalent annual investment costs:

$$W = V - \sum_{p=1}^{m} e_p \hat{k}_p \tag{1}$$

W, which we will call a "short-run cost function with continuously-adjusted capacity", is everywhere equal to the part of the objective function representing operating costs in the long-term program (i.e. the elementary function associated with operating costs). As W is equal to a difference of differentiable functions (assuming that the long-term solution is non-degenerate), W is itself differentiable in b. This comment is particularly important for the proofs of the propositions stated in the article.

2.4 Short-run marginal costs and marginal values of capacity

Where $L = \langle v, x\rangle + \langle y, b - Ax\rangle + \langle u, x_f - c - \hat{k}(b)\rangle$ denotes the usual Lagrangian constructed from the short-term program, we have:

$$W(b) = \max_{y,\, u \geq 0} \min_{x \in X} L(x, y, u, \hat{k}(b), b) \tag{2}$$

If we set $L(\bar{x}(y, u), y, u, \hat{k}(b), b) = \min_{x \in X} L(x, y, u, \hat{k}(b), b)$, the equation (2) can be rewritten as follows:

$$W(b) = \max_{y,\, u \geq 0} L(\bar{x}(y, u), y, u, \hat{k}(b), b) \tag{3}$$

Let $Y^*(b)$ denote the set of solutions (y, u) of the short-term dual program:

$$Y^*(b) = Arg \max_{y,\, u \geq 0} (\min_{x \in X} L(x, y, u, \hat{k}(b), b))$$

$Y^*(b)$ is not a singleton due to the degeneracy of the optimal solution (there are several vectors of dual variables and thus several saddle points). L thus possesses partial directional derivatives in b_i and each product has two short-run marginal costs. Where $s_i^+(b)$ denotes the right-hand short-run marginal cost of the product i and $s_i^-(b)$ its left-hand short-run marginal cost, we encounter the traditional results of the degeneracy (see for instance Milgrom and Segal 2002):

$$s_i^+(b) = \frac{\partial^+ L}{\partial b_i} = \max_{(y,\,u) \in Y^*(b)} \frac{\partial L}{\partial b_i} = \max_{(y,\,u) \in Y^*(b)} y_i$$

$$s_i^-(b) = \frac{\partial^- L}{\partial b_i} = \min_{(y,\,u) \in Y^*(b)} \frac{\partial L}{\partial b_i} = \min_{(y,\,u) \in Y^*(b)} y_i$$

Similarly, because of the degeneracy of the short-term problem, a left-hand marginal value and a right-hand marginal value are associated with each capacity constraint. As explained in the introduction, these marginal values are here negative: an increase in available capacity will not cause an increase in the operating cost.

For $p \in \{1, 2, ..., m\}$, let μ_p^- (respectively μ_p^+) be the left-hand (respectively right-hand) marginal value of the p^{th} capacity set at $c_p + \hat{k}_p(b)$ in the short-term problem.

We have: $|\mu_p^-| \geq e_p \geq |\mu_p^+|$

The proof is straightforward. For instance, let us assume that the second inequality is wrong. We then have $|\mu_p^-| > e_p$, which means that having an additional unit of available capacity generates a gain (i.e., a decrease in cost) higher than the cost of building this additional unit of capacity. This implies that the capacity is not adjusted in the long-run (which contradicts our assumptions).

More precisely, we have:

$$\mu_p^+ = \frac{\partial^+ L}{\partial \hat{k}_p} = \max_{(y,\,u) \in Y^*(b)} \frac{\partial L}{\partial \hat{k}_p} = \max_{(y,\,u) \in Y^*(b)} -u_p = -\min_{(y,\,u) \in Y^*(b)} u_p$$

$$\mu_p^- = \frac{\partial^- L}{\partial \hat{k}_p} = \min_{(y,\,u) \in Y^*(b)} \frac{\partial L}{\partial \hat{k}_p} = \min_{(y,\,u) \in Y^*(b)} -u_p = -\max_{(y,\,u) \in Y^*(b)} u_p$$

3 Specific case: model with a single capacity variable

3.1 Statement of the propositions

Our results can be summarized by the following propositions, stated for any given finished product:

(i) if its marginal capacity adjustment is positive, its right-hand (respectively left-hand) short-run marginal cost is equal to the difference between its marginal operating cost and the product of its marginal capacity adjustment by the left-hand (respectively right-hand) marginal value of capacity;

(ii) if its marginal capacity adjustment is negative, its right-hand (respectively left-hand) short-run marginal cost is equal to the difference between its marginal operating cost and the product of its marginal capacity adjustment by the right-hand (respectively left-hand) marginal value of capacity;

(iiia) the difference between its two short-run marginal costs is equal to the absolute value of the product of its marginal capacity adjustment by the difference between the left-hand and right-hand marginal values of capacity;

(iiib) if one considers two distinct finished products: the ratio of the differences between right-hand short-run marginal cost and left-hand short-run marginal cost is equal to the absolute value of the ratio of the marginal equivalent investment costs (or, which amounts to the same thing, to the absolute value of the ratio of the marginal capacity adjustments).

3.2 Proof of the propositions

As the long-term model includes only a single capacity variable, k, $\hat{k}(b)$, e, μ^- and μ^+ are scalars. A change in the demand for finished product i requires a marginal capacity adjustment equal to $\frac{\partial \hat{k}}{\partial b_i}\left(= \frac{\partial^+ \hat{k}}{\partial b_i} = \frac{\partial^- \hat{k}}{\partial b_i}\right)$ in order for the capacity to remain adjusted. The marginal equivalent investment cost is equal to $e\frac{\partial \hat{k}}{\partial b_i}$.

As the short-run cost function with continuously-adjusted capacity W is differentiable in b, its left-hand and right-hand partial derivatives with respect to b_i are equal. We will distinguish the two following cases.

3.2.1 A product with a positive marginal capacity adjustment

Consider a finished product i for which the marginal capacity adjustment is positive, i.e., $\frac{\partial \hat{k}}{\partial b_i} \geq 0$. By applying the envelope theorem to equation (3) we obtain the two following equations:

$$\frac{\partial W}{\partial b_i}(b) = \frac{\partial^+ W}{\partial b_i}(b) = \frac{\partial^+ L}{\partial b_i} + \frac{\partial^- L}{\partial \hat{k}} \frac{\partial^+ \hat{k}}{\partial b_i} = s_i^+(b) + \mu^- \frac{\partial \hat{k}}{\partial b_i} \quad (4)$$

$$\frac{\partial W}{\partial b_i}(b) = \frac{\partial^- W}{\partial b_i}(b) = \frac{\partial^- L}{\partial b_i} + \frac{\partial^- L}{\partial \hat{k}} \frac{\partial^- \hat{k}}{\partial b_i} = s_i^-(b) + \mu^- \frac{\partial \hat{k}}{\partial b_i} \quad (5)$$

Equations (4) and (5) can be analyzed as follows. First, since $\frac{\partial \hat{k}}{\partial b_i} \geq 0$, the capacity has to increase in order to remain adjusted when the demand for product i increases.

In equation (4), an increase in demand for product i on the right side of b_i implies that the change in the short-term program value caused by the capacity adjustment is computed using the left-hand marginal value of capacity. As the product has a positive marginal capacity adjustment, producing at the right-hand short-run marginal cost means that the available capacity is less than the adjusted one, which explains why the left-hand marginal value of capacity is used to value the capacity adjustment (in other words, the adjusted capacity is attained from below).

Conversely, in equation (5), an increase in demand for product i on the left side of b_i implies that the change in the short-term program value is computed using the right-hand marginal value of capacity. Here, producing at the left-hand short-run marginal cost means implicitly that the available capacity is greater than the adjusted one. Consequently, the adjusted capacity is attained from above, which justifies using the right-hand marginal value of capacity.

Equations (4) and (5) thus give us:

$$\frac{\partial W}{\partial b_i}(b) = s_i^{+}(b) + \mu^{-}\frac{\partial \hat{k}}{\partial b_i} = s_i^{-}(b) + \mu^{+}\frac{\partial \hat{k}}{\partial b_i} \tag{6}$$

Furthermore, by differentiating equation (1) we obtain:

$$\frac{\partial W}{\partial b_i}(b) = \frac{\partial V}{\partial b_i}(b) - e\frac{\partial \hat{k}}{\partial b_i}(b) = l_i(b) - e\frac{\partial \hat{k}}{\partial b_i}(b) \tag{7}$$

Equation (7) indicates that $\frac{\partial W}{\partial b_i}(b)$ is equal to the marginal operating cost (i.e., the long-run marginal cost less the marginal equivalent investment cost).

By combining equations (6) and (7), we obtain:

$$l_i(b) = s_i^{+}(b) + (\mu^{-} + e)\frac{\partial \hat{k}}{\partial b_i} = s_i^{-}(b) + (\mu^{+} + e)\frac{\partial \hat{k}}{\partial b_i}$$

This result is crucial since it explains the transition from the long-run marginal cost to short-run marginal costs. And finally:

$$s_i^{+}(b) = \left(l_i(b) - e\frac{\partial \hat{k}}{\partial b_i}\right) - \mu^{-}\frac{\partial \hat{k}}{\partial b_i} \tag{8}$$

$$s_i^{-}(b) = \left(l_i(b) - e\frac{\partial \hat{k}}{\partial b_i}\right) - \mu^{+}\frac{\partial \hat{k}}{\partial b_i} \tag{9}$$

Equations (8) and (9) proves the proposition (i).

3.2.2 A product with a negative marginal capacity adjustment

Now consider a product j for which the marginal capacity adjustment is negative, i.e., $\frac{\partial \hat{k}}{\partial b_j} \leq 0$. By applying the envelope theorem to equation (3), we obtain:

$$\frac{\partial W}{\partial b_j}(b) = \frac{\partial^+ W}{\partial b_j}(b) = \frac{\partial^- L}{\partial b_j} + \frac{\partial^+ L}{\partial \hat{k}} \frac{\partial^+ \hat{k}}{\partial b_j} = s_j^+(b) + \mu^- \frac{\partial \hat{k}}{\partial b_j} \tag{10}$$

$$\frac{\partial W}{\partial b_j}(b) = \frac{\partial W}{\partial b_j}(b) = \frac{\partial^- L}{\partial b_j} + \frac{\partial^- L}{\partial \hat{k}} \frac{\partial^- \hat{k}}{\partial b_j} = s_j^-(b) + \mu^- \frac{\partial \hat{k}}{\partial b_j} \tag{11}$$

The interpretation of equations (10) and (11) is similar to that of equations (5) and (4). For instance, as the marginal capacity adjustment of the product considered is negative, producing at the right-hand short-run marginal cost implicitly means that the available capacity is greater than the adjusted one, whence the valuation of the capacity adjustment at the right-hand marginal value of capacity in equation (10). By proceeding as in the earlier instance, we finally obtain via equations (7), (10) and (11):

$$s_j^+(b) = \left(l_j(b) - e\frac{\partial \hat{k}}{\partial b_j}\right) - \mu^+ \frac{\partial \hat{k}}{\partial b_j} \tag{12}$$

$$s_j^-(b) = \left(l_j(b) - e\frac{\partial \hat{k}}{\partial b_j}\right) - \mu^- \frac{\partial \hat{k}}{\partial b_j} \tag{13}$$

Equations (12) and (13) prove the proposition (ii).

3.2.3 Propositions (iiia) and (iiib)

For any finished product i, the proposition (iiia) is immediately deduced from the preceding equations:

$$s_i^+(b) - s_i^-(b) = \left|(\mu^- - \mu^+)\frac{\partial \hat{k}}{\partial b_i}\right|$$

For a given finished product, the difference between the two short-run marginal costs is therefore proportional to the marginal equivalent investment cost assigned to this product. We do in fact have for any product i: $s_i^-(b) \leq l_i(b) \leq s_i^+(b)$

If we consider two finished products i and j, we have:

$$\frac{s_i^+(b) - s_i^-(b)}{s_j^+(b) - s_j^-(b)} = \left|\frac{\partial \hat{k}/\partial b_i}{\partial \hat{k}/\partial b_j}\right| = \left|\frac{\text{marginal equivalent investment cost of product } i}{\text{marginal equivalent investment cost of product } j}\right|$$

This equation proves the proposition (iiib): the ratio of the differences between short-run marginal costs is equal to the absolute value of the ratio of the marginal equivalent investment costs (i.e., the ratio of the marginal capacity adjustments).

4 General case: model with several capacity variables

4.1 Statement of the propositions

Let us consider a given finished product i. Let Z_i^+ denote the set of capacity variables for which the marginal equivalent investment cost associated with product i is positive. Z_i^- then denotes the set of capacity variables for which the marginal equivalent investment cost associated with product i is negative. We can state the following propositions:

(j) the left-hand short-run marginal cost is given by the following formula:

$$\text{marginal operating cost} - \sum_{p \in Z_i^+} \left(\mu_p^+ \frac{\partial \hat{k}_p}{\partial b_i}\right) - \sum_{p \in Z_i^-} \left(\mu_p^- \frac{\partial \hat{k}_p}{\partial b_i}\right)$$

(jj) the right-hand short-run marginal cost is given by the following formula:

$$\text{marginal operating cost} - \sum_{p \in Z_i^+} \left(\mu_p^- \frac{\partial \hat{k}_p}{\partial b_i}\right) - \sum_{p \in Z_i^-} \left(\mu_p^+ \frac{\partial \hat{k}_p}{\partial b_i}\right)$$

(jjj) the difference between the two short-run marginal costs is equal to:

$$\sum_{p=1}^{m} \left| (\mu_p^- - \mu_p^+) \frac{\partial \hat{k}_p}{\partial b_i} \right|$$

4.2 Proof of the propositions

The long-run marginal cost of a given finished product i is broken down into the sum of $m + 1$ terms:

- the marginal operating cost,
- m marginal equivalent investment costs.

Without loss of generality, we will assume that the marginal equivalent investment costs corresponding to the first z capacity variables are positive and that those corresponding to the other $m - z$ are negative. In other terms, the set of the first z capacity variables forms Z_i^+. The marginal capacity adjustments are as follows:

$$\frac{\partial \hat{k}_p}{\partial b_i}(b) \geq 0 \text{ for } p \in \{1, 2, ..., z\}$$

$$\frac{\partial \hat{k}_p}{\partial b_i}(b) \leq 0 \text{ for } p \in \{z+1, z+2, ..., m\}$$

Following the same line of reasoning as was used in the specific instance of a single capacity constraint, we obtain:

$$\frac{\partial W}{\partial b_i}(b) = s_i^+(b) + \sum_{p=1}^{z} \mu_p^- \frac{\partial \hat{k}_p}{\partial b_i} + \sum_{p=z+1}^{m} \mu_p^+ \frac{\partial \hat{k}_p}{\partial b_i} = s_i^-(b) + \sum_{p=1}^{z} \mu_p^+ \frac{\partial \hat{k}_p}{\partial b_i} + \sum_{p=z+1}^{m} \mu_p^- \frac{\partial \hat{k}_p}{\partial b_i} \quad (14)$$

Moreover, by differentiating equation (1) we have:

$$\frac{\partial W}{\partial b_i}(b) = l_i(b) - \sum_{p=1}^{m} e_p \frac{\partial \hat{k}_p}{\partial b_i}(b) \quad (15)$$

As previously, $\frac{\partial W}{\partial b_i}(b)$ is equal to the marginal operating cost.
By combining equations (14) and (15), we obtain:

$$s_i^-(b) = \left(l_i(b) - \sum_{p=1}^{m} e_p \frac{\partial \hat{k}_p}{\partial b_i}\right) - \sum_{p \in Z_i^+} \left(\mu_p^+ \frac{\partial \hat{k}_p}{\partial b_i}\right) - \sum_{p \in Z_i^-} \left(\mu_p^- \frac{\partial \hat{k}_p}{\partial b_i}\right) \quad (16)$$

$$s_i^+(b) = \left(l_i(b) - \sum_{p=1}^{m} e_p \frac{\partial \hat{k}_p}{\partial b_i}\right) - \sum_{p \in Z_i^+} \left(\mu_p^- \frac{\partial \hat{k}_p}{\partial b_i}\right) - \sum_{p \in Z_i^-} \left(\mu_p^+ \frac{\partial \hat{k}_p}{\partial b_i}\right) \quad (17)$$

Equations (16) and (17) prove the propositions (j) and (jj). By combining these two equations, we easily prove the proposition (jjj).

5 Application to a simplified refining model

We will use the refining model developed by Pierru and Babusiaux (2004) to illustrate their methodology for the breakdown of long-run marginal costs. The authors present a simplified model of a refinery for which a catalytic cracking unit must be built in order to increase gasoline yield (four finished products are produced: gasoline, automotive diesel, heating oil and heavy fuel oil). At the optimum, the long-run marginal costs and their breakdown are given in Table 1.

Oil product	Gasoline	Automotive diesel	Heating oil	Heavy fuel oil
Marginal equivalent investment cost	35.5	-9.4	-12.8	0
Marginal operating cost	209.3	183.7	172.1	0
Long-run marginal cost	244.8	174.3	159.3	0

Table 1: *Long-run marginal costs per oil product ($)*

We will now determine the short-run marginal costs for each product. To do so we must convert the long-term model into a short-term model.

Only the operating cost (denoted f_1 in the original article) now appears in the objective cost function to be minimized. As the installed capacity of the cracking unit (determined at the optimum of the long-term model) is equal to 1010372.30 tons, the following constraint must be added (k being notated CAP in the original article):

$$CAP \leq 1{,}010.4$$

This constraint specifies that, in the short run, the quantity of distillate (expressed in thousands of tons) processed by the cracking unit cannot exceed the installed capacity. For a given product, an increase in the quantity produced leads to an optimal basic solution different from the solution for a decrease in the quantity produced.

In the example, e is taken as equal to $28 per ton. If one decreases the right-hand-side coefficient of the capacity constraint by one ton in the short-run model, the objective cost function increases by $376.18. We therefore have: $\mu^- = -376.18$. If one increases this coefficient by one ton, the value of the short-term objective function remains unchanged. We therefore have here $\mu^+ = 0$.

Table 2 shows the short-run marginal costs obtained. Three values - shown in boldface – are equal to marginal operating costs previously determined with the long-term model: the left-hand short-run marginal cost of the gasoline and the right-hand short-run marginal cost of the automotive diesel and heating-oil. The three other short-run marginal costs are different. This result is consistent with propositions (i) and (ii) as here the right-hand marginal value of capacity is equal to zero.

If we consider the gasoline, for example, it presents a positive marginal equivalent investment cost. Consequently, in the short run, producing a ton less of gasoline implies here that all available capacity is not used. The capacity constraint is thus no longer binding, and the new basic solution is the same as that considered at the optimum in the long-term analysis (with the same binding equations). The left-hand short-run marginal cost is therefore equal to the marginal operating cost. Conversely, if an additional ton of gasoline must be produced, the capacity constraint remains binding (due to the fact that the marginal equivalent investment cost is positive) and another constraint of the model becomes inactive. The new basic solution is therefore no longer the one determined at the optimum of the long-term model, and the right-hand short-run marginal cost of gasoline is consequently different from the marginal operating cost.

If we consider automotive diesel (or heating oil), the reasoning is identical but functions in reverse order. Since the marginal equivalent investment cost is negative, producing an additional ton implies that the full available capacity of the cracker is no longer used. Since the capacity constraint is no longer binding, the new basic solution is implicitly the same as that determined at the optimum of the long-term analysis. The right-hand short-run marginal cost is thus equal to the marginal operating cost. Conversely, decreasing the production implies binding the capacity constraint. Another constraint thus becomes inactive and the basic solution is different from that determined for the long-term. For this reason, the left-hand short-run marginal cost is different from the marginal operating cost.

	Left-hand short-run marginal cost	**Long-run marginal cost**	**Right-hand short-run marginal cost**
Gasoline	**209.3**	244.8	686.4
Automotive diesel	57.2	174.3	**183.7**
Heating oil	0	159.3	**172.1**

Table 2: *Left-hand and right-hand short-run marginal costs ($)*

Proposition (iiib) is also verified. If we consider gasoline and automotive diesel fuel, for instance, we have:

$$\frac{686.4-209.3}{183.7-57.2} = \left|-\frac{35.5}{9.4}\right|$$

The ratio between the marginal equivalent investment cost of gasoline and that of automotive diesel shows us that the ratio of the differences of short-run marginal costs is equal to 3.77. The marginal equivalent investment costs thus enable us to know the relative amplitude of the "jumps" in short-run marginal costs.

Property (iiia) is satisfied for all finished products. If we take the example of gasoline, we have (rounded off):

$$686.4-209.3 = \frac{35.5\times(376.18-0)}{28}$$

6 Conclusion: economic interpretation

In conclusion, we will emphasize the economic interpretation of the results obtained, formulated by the propositions in sections 3 and 4. We will first consider the specific case of a model with only one capacity variable.

Take the example of a finished product for which the marginal equivalent investment cost is positive. The right-hand short-run marginal cost of this product is given by the right-hand term of equation (8) which represents the sum of two terms. The first term is equal to the marginal operating cost (i.e., the cost of the optimal change over the long run in inputs which remain variable over the short run). The second term is equal to the marginal capacity adjustment (i.e., the additional capacity required in the long run to produce an additional unit of the finished product) multiplied by the left-hand marginal value of the capacity in the short run. Since one cannot have this additional capacity in the short term, a cost premium, equal to this additional capacity multiplied by the left-hand marginal value of the capacity, is generated. Conversely, the left-hand short-run marginal cost of this product is equal to the marginal operating cost less the marginal capacity adjustment multiplied by the right-hand marginal value of capacity. In fact, in the long run, increasing the production of the finished product, to the point of meeting anticipated demand, requires having additional capacity. In the short run, however, this additional capacity is available (since the capacity was set in relation to anticipated demand). As a result, the cost premium generated is computed using the right-hand marginal value of capacity.

A similar analysis can be made for a product with a negative marginal equivalent investment cost. In this case, both short-run marginal costs are less than the marginal operating cost since, in the short run, producing more entails a relaxation of the capacity constraint thereby saving cost.

It is interesting to note that in the specific case in which the absolute value of the right-hand (or left-hand) marginal value of capacity is equal to the equivalent annual investment cost per unit of installed capacity, each finished product has a short-run marginal cost equal to the long-run marginal cost.

This type of analysis is also applicable in the general case of models with several capacity variables. Each short-run marginal cost is equal to the marginal operating cost increased by a sum of additional costs incurred and decreased by a sum of costs saved.

In addition to their theoretical interest, these results, which complement traditional microeconomic theory, can be quite useful in practice. There are numerous constraints and variables in the field of oil refining. Apart from extreme situations, the number of steps comprising marginal cost curves is such that the curves are "smoothed" and are not far removed from those of traditional microeconomic theory. However, for runs with highly-binding capacities, the "jumps" in marginal costs can be significant. This is all the more true for peak demand in the electricity sector, where power cannot be stockpiled. A better comprehension of marginal costs should thus serve as a valuable aid for market power analysis.

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