Efficient procurement with quality concerns*

Pierre-Henri Morand
Lionel Thomas
CRESE, Université de Franche-Comté, France**

1 Introduction

In many countries, intensive purchasing makes the government a major or dominant buyer of a wide range of goods and services. Most of the time, numerous quality considerations are incorporated into the contract and the quality level offered by sellers directly affects the social value associated with the project. Hence, most procurement settings have in common that the procurer cares about qualitative attributes of the goods (the contract) he wishes to buy. The procurement requirements are therefore those procuring entities not only evaluate tenders on the basis of the price, but of the greater economic advantage. Besides, the public acquisition regulation laws generally impose to take quality considerations explicitly into account but also to use as much as possible a competitive procurement process. As for example, the U.S. Federal Acquisition Rule “requires [...] that contracting officers shall promote and provide for full and open competition in soliciting offers and awarding Government contracts” (FAR 6.101). Similar concerns about qualitative attributes of the goods (the contract) a firm wishes to buy can be founded in corporate procurement. Traditionally, these types of negotiations are resolved through bilateral bargaining. But an increasing part of these contracts are procured via specific forms of multi-attribute auction. Multi-attribute procurement auctions usually require the bid to specify several characteristics of the contract to be fulfilled, and are procured via a Request for Quotes (RFQ) process. An RFQ process allows the

* The authors are very grateful to Mark Armstrong and anonymous referees. Usual disclaimers apply.
** Université de Franche Comté, UFR SJEPC, 45D avenue de l’Observatoire, 25030 Besançon cedex, France.
sale to be determined by a variety of attributes, involving not only price, but quality, lead time, contract terms, supplier reputation, and incumbent switching costs... The predominant approach in RFQ practice is for the auctioneer to announce a scoring rule in terms of the bid price and the various qualitative attributes. Recently, with the development of eMarketplaces, multi-attribute reverse auctions have become a popular means of automating this process further in corporate procurement. The negotiable attributes are defined in advance, and suppliers can compete either in an open-cry or sealed-bid fashion on multiple attributes.

It is useful to highlight distinct types of quality, which may be found in procurement contract. Most of the models of procurement assume that quality is either fixed (as in Myerson (1981)) or contractible (as in Che (1993), Branco (1997)). This implies that the sellers can adapt their level of quality supplied to a specific requirement. But, in many cases, these attributes are determined by prior sellers’ investments or choice of technology. Quality becomes therefore an intrinsic parameter of the seller, which can only offer a single fixed level of quality. This paper deals with this latter kind of quality. Furthermore, many procurement contracts are characterized by an asymmetry in the information available to the buyer on the quality of the product. Sellers know what they are selling, while buyers often do not know what they are buying.\(^1\) Quality becomes observable by the buyer after contract completion. But observable quality is not necessarily at odds with verifiable quality, i.e. its level can be described ex ante in a contract and ascertained ex post by a court. This paper is concerned with intrinsic and observable, but unverifiable and therefore non-contractible quality.\(^2\)

Hence, we design the optimal procurement mechanisms, when sellers are privately informed on efficiency and on observable but neither verifiable nor contractible quality. We consider not only that efficiency reduces the firm’s cost to produce a given quality level but also that a high quality contract is more costly for firms and more valuable for buyer than a low quality. So we have a common value model because one dimension of the private information, namely quality, both affects firms’ costs and buyer’s gross surplus.

We show that most of the optimal procurement institutions are mixed procedure implying both separation and pooling. Indeed, the buyer faces a conflict between his own incentives (surplus) which tend to favor high quality contracts and private incentives, favoring low cost firms.\(^3\) Since high quality is preferred but costly, selecting high quality firms impedes private

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1 Alternative assumption can be found in Rezende (2004), where “the costs are the suppliers’ private information; the values are the buyer’s private information”.

2 There is a more complex case corresponding to our setting, that of trust or credence goods, which refers to situations where consumers cannot possibly know the characteristics of a product after consumption even if the actual level of quality affects the real consumer surplus. Such examples can be found in the medical care market.

3 A similar argument can be found in the Armstrong and Vickers (2000) analysis of multiproduct monopoly, where firm’s private incentives can not be necessarily aligned to socially desirable outcomes.
incentives because it leads the low quality firm to tender non-truthfully. Nevertheless, the presence of the efficiency parameter mitigates this conflict, because buyer's preferences on efficiency and private incentives work in the same way. So, the buyer must choose the optimal trade-off between private incentives it must conceal to firms and its own incentives by increasing the surplus he can obtain from the procurement contract. It follows that it is optimal to design procedures which bunch certain types and separate others. So a procurement auction (a fully separating contract) is optimal only if the buyer's incentives and private incentives are commononotic. On the other hand, the optimal procurement rules implies a take-it-or-leave-it offer (a fully pooling contract) only if the buyer's incentives always favor the high quality regardless of the efficiency.

From an applied economic point of view, our model highlights some drawbacks of widely used competitive process for allocating multi-attribute contracts. It will present conditions under which seeking bids or randomly select supplier is optimal. We show that most of these procedures may imply some partial pooling of types, which is very far from actual practice or public procurement legislations, which either imply a totally separating mechanism when an auction is used or a totally pooling mechanism when the buyer uses a take-it-or-leave-it offer.

This conflict between buyer's incentives which tend to favor high quality contracts and private incentives is known under the term non-responsiveness that appears in common value models. Following Morand and Thomas (2003), we know that, contrarily to the results of most of the models with non-responsiveness, separation may occur even when the conflict exists. In a procurement setting, this phenomenon affects the allocative efficiency of the mechanism, according to the impact of information rents on buyer's preferences. We distinguish neutral rents (i.e. rents such that sellers' types with incomplete information are ordered exactly as with complete information) from non-neutral ones. It follows that if rents are neutral, then bunching precludes efficiency, and a more separating procedure is, as traditionally, a more efficient one. On the other hand, if rents are non-neutral, then a mixed procedure implies less bunching, but, counter-intuitively, such separation increase does not necessarily enhance allocative efficiency: the buyer is incited to select a different seller than with complete information.

Our model departs from the traditional optimal procurement literature results. These latters either establish that, under wide conditions, an auction process is optimal (Myerson (1981), Che (1993)) or provide conditions under which the take-it-or-leave-it offer is the best procedure (Manelli and Vincent (1995)). More precisely, in the former, quality is either fixed or contractible so that only firms' efficiency matters in the relationship. Then, under symmetry of sellers and regularity of prior beliefs, the maximizing buyer's surplus procedure is a procurement auction. That is, a fully separating contract is optimal that specifies a different expected probability of winning for each type. Moreover, the procedure is efficient since the asymmetry of information does not modify the buyer's preferences: the same seller
than with complete information has to be selected.\footnote{See also Branco (1997) when bidder’s efficiency is correlated.} In the latter, where cost only depends on unverifiable quality, the authors first suggested that take-it-or-leave-it offers rather than procurement auction might be optimal and do not address the question of efficiency. These procedures appear as polar cases of the more general mixed procedure depicted here, where both efficiency and quality are private informations.

Our paper is also related to the multidimensional screening literature. Most of this literature designs the mechanisms when the principal can screen over the agent using as many instruments as the dimensionality of the vector of private information.\footnote{See e.g. Rochet and Choné (1998) and Armstrong and Rochet (1999).} The optimal multi-unit auction is the typical auction application: a seller wants to maximize his expected revenue from the sale of several goods determining optimal probabilities of selection (the instruments) for each goods. Buyers’ private information corresponds to their valuations for each good (Armstrong (2000), Avery and Henderschott (2000), \ldots). A central question that these authors address is to what extent the auctioneer may benefit from bundling the objects. We depart from these previous multidimensional analysis given that only one good is traded, namely the contract, which can only be allocated to one seller. Moreover, both private informations affect the cost of producing the contract. We have here just one instrument (the probability of awarding the contract) to screen over two types (efficiency and quality). Hence, none of the existing characterization results applies to our problem.

The remainder of the paper is organized as follows. Section 2 lays out the general framework of the model and derives some incentive compatibility conditions. Section 3 looks at optimal mechanisms first with complete information, then with incomplete information, sheds light on the efficiency properties of the optimal mechanisms and proposes an implementation of the optimal solutions, using a specific scoring rule. Section 4 provides an extension of our model, assuming alternative assumptions. Finally, unless included in the text, most of the proofs are provided in the Appendix.

## 2 The model

A risk-neutral buyer wishes to contract with a single seller in order to purchase an indivisible good. There are \( n \) risk-neutral potential sellers competing for the contract. This contract can be carried out with high or low quality. Let \( q_2 \) denote the high quality parameter and \( q_1 \) the low one. Quality directly affects the buyer’s valuation of the contract. Let \( S(q_i) = S_i \) be the gross surplus of contract carried out with quality \( i = 1, 2 \). We have:

\[
S_2 > S_1
\]
Let $\Delta S = S_2 - S_1$. Each firm privately knows the cost it is able to undertake to deliver the good. The firm's cost depends on the intrinsic quality and efficiency. These two parameters are assumed to be unobservable by the buyer and independently distributed across sellers. Let $\theta_1$ denote the higher efficiency parameter and $\theta_2$ the lower.\footnote{Clearly, assuming a discrete 2x2 model simplifies the computations and enables to focus on the intuition. But, we adopt this framework for two other reasons: it avoids bunching which inevitably happens with two dimensions of continuously distributed private information and only one non-monetary instrument (i.e.: the probability) as in Laffont et alii (1987) and exclusion of a non-empty set of types as in Armstrong (1999).} The cost of a firm carrying out quality $i = 1, 2$ with efficiency $j = 1, 2$ is:

$$c(q_i, \theta_j) = c_{ij}$$

We assume that supplying a greater quality is costly whereas supplying a better efficiency reduces cost. So, we obtain:

$$c_{2j} > c_{1j} \ \forall j = 1, 2$$

$$c_{12} > c_{11} \ \forall i = 1, 2$$

To complete the cost ordering, it remains an ambiguity between $c_{21}$ and $c_{12}$. We assume in most of the developments that (the alternative assumption is made in section 4):

$$c_{21} < c_{12}$$

Intuitively, it is assumed here that is is always less costly to supply the contract with a high efficiency. It follows that cost is completely ordered as:\footnote{Types are ordered in the sense of Rochet and Stole (2003). More precisely, one key technical difficulty in most multidimensional screening problems is the endogenous nature of the ordering agent's type. Here, because both private information affect production cost (from the seller point of view), the ordering agent's type is exogenously given.}

$$c_{11} < c_{21} < c_{12} < c_{22}$$

Besides, we assume that:

A.1: $c_{21} - c_{11} \leq c_{12} - c_{21} \leq c_{22} - c_{12}$

So, cost increases between adjacent types are non-decreasing. It implies, especially, that an increase in quality is all the less costly that efficiency is high. Finally, let $\pi_{ij}$ be the a priori probability that a firm has type $c_{ij}$. We assume that this prior satisfies the monotone hazard rate. That is:

A.2: $\frac{\pi_{11}}{\pi_{21}} \leq \frac{\pi_{11} + \pi_{21}}{\pi_{12}} \leq \frac{\pi_{11} + \pi_{21} + \pi_{12}}{\pi_{22}}$

A.1 and A.2 together lead to the standard regulatory condition assumed in most of the models of the screening literature.
2.1 The reduced form probabilities modelling and the feasibility constraints

By the Revelation Principle, there is no loss of generality in restricting our attention to incentive compatible direct mechanisms. Adopting a reduced form probabilities modeling (Border (1991), Armstrong (2000)), the selection and the payment rules are determined as functions only of the sellers’ reported types and not to all the types of all sellers. Then an optimal procurement mechanism can be summarized by eight functions:

\[ \{p_{11}, p_{12}, p_{21}, p_{22}, T_{11}, T_{12}, T_{21}, T_{22}, \} \]

where \( p_{ij} \) denotes the expected probability that a firm with type \( ij \) obtains the contract and \( T_{ij} \) the expected payment to the firm.

Let \( R_{ij} = T_{ij} - p_{ij}c_{ij} \) be the expected rent of a firm reporting its private information truthfully. The buyer wishes to maximize the following expected net surplus function:

\[
E(W) = n \left[ \sum_{i,j} \pi_{ij} (p_{ij}S_i - T_{ij}) \right]
\]

which can be rewritten:

\[
E(W) = n \left[ \sum_{i,j} \pi_{ij}p_{ij}(S_i - c_{ij}) - \sum_{i,j} \pi_{ij}R_{ij} \right]
\]

(1)

We need to consider the following constraints:

- the mechanism is incentive compatible if each seller reports his private information truthfully. Since for each type \( ij \), the alternative report \( i'j' \) yields an expected rent of \( T_{i'j'} - p_{i'j'}c_{ij} = R_{i'j'} - p_{i'j'}(c_{ij} - c_{i'j'}) \), we define the 12 incentive constraints:

\[
R_{ij} \geq R_{i'j'} - p_{i'j'}(c_{ij} - c_{i'j'}), \ \forall i, j = 1, 2 \ \forall i', j' = 1, 2
\]

(2)

- the mechanism is individually rational if all sellers are better-off participating in the mechanism. We need:

\[
R_{ij} \geq 0 \ \forall i, j = 1, 2
\]

(3)

- the mechanism is feasible if reduced form probabilities satisfy feasibility constraints. Following Armstrong (2000), let \( \left( \sum_{ij \in S} \pi_{ij} \right)^n \) be the probability that all the sellers are in a given set \( S, \forall S \subseteq \{11, 12, 21, 22\} \). Thus, \( 1 - \left( 1 - \sum_{ij \in S} \pi_{ij} \right)^n \) is the probability that there is at least one seller in
this set. From (1), \( n \sum_{ij \in S} \pi_{ij} p_{ij} \) is the probability that a seller from a
given set of types wins. Therefore, the feasibility requires to satisfy the 15
following constraints:

\[
\sum_{ij \in S} \pi_{ij} p_{ij} \leq 1 - \left( 1 - \sum_{ij \in S} \pi_{ij} \right)^n, \quad \forall S \subseteq \{11, 12, 21, 22\} \tag{4}
\]

that is, the probability that a seller from a given set of types wins must be
less or equal to the probability that there is at least one seller in this set.
The buyer's problem \((P)\) becomes:

\[
\max_{(p_{ij}, R_{ij})} E(W) \text{ subject to (2), (3) and (4)}
\]

For simplicity, we suppose that the gross surplus is sufficiently high that it
is worth producing for any \( c_{ij} \), and for any quality level \( q_i \).

### 2.2 The incentive constraints and the monotonicity condition

Following standard developments in mono or multidimensional incentives
theory (see e.g. Laffont and Martimort (2002)), we first solve a relaxed pro-
blem involving only a subset of incentive constraints, and checking ex post
the remaining constraints. It enables us to rewrite the IC and IR constraints
into a more tractable form. Hence, we begin by deriving from (2) the fol-
lowing ordering on the expected probabilities (the details of the calculus are
contained in the Appendix A):

\[
p_{11} \geq p_{21} \geq p_{12} \geq p_{22} \tag{5}
\]

Recall that cost is ordered as \( c_{11} < c_{21} < c_{12} < c_{22} \). So the expected
probability must be even higher that the cost is low. This suggests that
the three binding incentive constraints are such that a 11-type must not
be tempted to choose the 21-type contract, a 21-type to choose 12-type, a
12-type to choose 22-type. Clearly, a 11-type which chooses directly a 22-
type contract would not benefit from rents that he can obtain by mimicking
type 21 then 12 then 22. To rely on incentives theory litterature, it yields
that upward incentives constraints are binding.\(^8\) It follows that rents can be
rewritten as a function of expected probabilities in order to reformulate
the objective function. We can easily show that if \( R_{22} \geq 0 \) is satisfied
then individual rationality is satisfied for all other types. With an objective
function decreasing in \( R_{ij} \), the set of individual rationality constraints (3)
implies:

\[
R_{22} = 0
\]

\(^8\) So in appendix A, constraints (10), (12) and (15) bind.
Assuming the latter participation and upward incentive constraints bind, we have:

\[ R_{22} = 0 \]
\[ R_{12} = (c_{22} - c_{12})p_{22} \]
\[ R_{21} = (c_{22} - c_{12})p_{22} + (c_{12} - c_{21})p_{12} \]
\[ R_{11} = (c_{22} - c_{12})p_{22} + (c_{12} - c_{21})p_{12} + (c_{21} - c_{11})p_{21} \]

It follows that expected rents correspond to informational rents. So the objective function becomes after collecting terms:

\[ E(W) = n \left\{ \pi_{11}p_{11}(S_1 - c_{11}) + \pi_{21}p_{21}\left(S_2 - c_{21} - \frac{\pi_{11}}{\pi_{21}}(c_{21} - c_{11})\right) + \pi_{12}p_{12}\left(S_1 - c_{12} - \frac{\pi_{11} + \pi_{21}}{\pi_{12}}(c_{12} - c_{21})\right) + \pi_{22}p_{22}\left(S_2 - c_{22} - \frac{\pi_{11} + \pi_{21} + \pi_{12}}{\pi_{22}}(c_{22} - c_{12})\right) \right\} \]

Hence, except for the lowest cost, informational rents increase the cost for the buyer. Let \( C_{ij} \) and \( W_{ij} \) be respectively the virtual cost (i.e. cost plus informational rent) and the virtual surplus (i.e. social surplus minus informational rent) associated with the \( ij \) type, we obtain:

\[ E(W) = n \left\{ \pi_{11}p_{11}(S_1 - C_{11}) + \pi_{21}p_{21}(S_2 - C_{21}) + \pi_{12}p_{12}(S_1 - C_{12}) + \pi_{22}p_{22}(S_2 - C_{22}) \right\} \]

\[ = n \sum_{i,j} \pi_{ij}p_{ij}W_{ij} \]

Under the regularity assumptions A.1, and A.2, we obtain that virtual costs are ordered exactly as costs. Finally, the relaxed problem becomes:

\[ \max_{p_{ij}} E(W) = n \sum_{i,j} \pi_{ij}p_{ij}W_{ij} \text{ s.t. } (4), (5) \]

The following section focuses on the design of the optimal procurement mechanisms.

### 3 Optimal procurement mechanisms

We begin this section by suggesting the intuition of the solution which leads us to formulate simplifying assumptions for the next developments.
3.1 A heuristic description of the solution

By examining the problem of the buyer, we easily observe that it is linear with respect to $p_{ij}$. So, the problem takes two steps: first the buyer has to seek for the highest virtual surplus $W_{ij}$, which tends him to associate a higher expected probability with a higher virtual surplus. Second, he must confront these expected probabilities to the constraint (5) to comply with incentives. Thus, to complete the first task, he has to order the virtual surpluses, comparing quality and virtual costs they induce. For example: a 22-type yields a gross surplus of $S_2$ and a virtual cost of $C_{22}$, while a 12-type yields a gross surplus of $S_1$ and a virtual cost of $C_{12}$. So, 22-types are preferred if:

$$S_2 - C_{22} > S_1 - C_{12}$$
$$\Leftrightarrow \Delta S > C_{22} - C_{12}$$

$$\Leftrightarrow \Delta S > c_{22} + \frac{\pi_{11} + \pi_{21} + \pi_{12}}{\pi_{22}}(c_{22} - c_{12}) - \left[c_{12} + \frac{\pi_{11} + \pi_{21}}{\pi_{12}}(c_{12} - c_{21})\right] \quad (6)$$

that is, if the difference in quality dominates the difference in virtual cost. Since the virtual costs take a very general form, ordering all virtual surpluses becomes rapidly a tedious task. In order to simplify the first step of the calculus and to focus on intuitions, we make the following assumptions that satisfy A.1 and A.2:

$$\Delta c = c_{22} - c_{12} = c_{12} - c_{21} = c_{21} - c_{11}$$

so that, cost increases between adjacent types are assumed to be constant and:

$$\pi_{ij} = \pi, \forall i, j = 1, 2$$

that is, the prior is uniform. Then (6) reduces to:

$$\Delta S > 2\Delta c$$

More precisely, the first step of the buyer’s problem consists in framing the value of $\Delta S$ by $\Delta c$ multiples. This principle is applied to the first-best solution, then to the incomplete information framework.

3.2 Complete information solution

To begin, we briefly present the complete information setting in order to determine the first-best efficient benchmarks. To this end, let $w_{ij} = S_i - c_{ij}$ represents the social surplus. Hence, to each type of seller corresponds a specific level of social surplus. The complete information problem is so:

$$\max_{p_{ij}} W = n \sum_{i,j} \pi_{ij} p_{ij} w_{ij} \text{ s.t. } (4)$$
Roughly speaking, since the buyer observes types, he has no incentives problem, then nor rents to concede. Thus, he only has to seek for the highest social surplus. Using previous assumptions, we can derive three distinct social surplus orderings (hereafter SSO):

- **SSO 1**: If \( \Delta S < \Delta c \), then the ordering is \( w_{11} > w_{21} > w_{12} > w_{22} \). The quality is never socially desirable.
- **SSO 2**: If \( \Delta c < \Delta S < 3 \Delta c \), then the ordering is \( w_{21} > w_{11} > w_{22} > w_{12} \). The quality is desirable whichever the efficiency, but not between the highest and the lowest costs.
- **SSO 3**: If \( \Delta S > 3 \Delta c \), then the ordering is \( w_{21} > w_{22} > w_{11} > w_{12} \). The quality is always socially desirable.

This complete information benchmark enables us to define three first-best optimal mechanisms:

**Lemma 1** With complete information, the optimal procurement selection rules are within SSO:

- **1**: \( p_{11}^{fb} = \frac{1 - (1 - \pi)^n}{n \pi} \), \( p_{21}^{fb} = \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n \pi} \),
  \( p_{12}^{fb} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n \pi} \), \( p_{22}^{fb} = \frac{\pi^n}{n \pi} \)

- **2**: \( p_{21}^{fb} = \frac{1 - (1 - \pi)^n}{n \pi} \), \( p_{11}^{fb} = \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n \pi} \),
  \( p_{22}^{fb} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n \pi} \), \( p_{12}^{fb} = \frac{\pi^n}{n \pi} \)

- **3**: \( p_{21}^{fb} = \frac{1 - (1 - \pi)^n}{n \pi} \), \( p_{22}^{fb} = \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n \pi} \),
  \( p_{11}^{fb} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n \pi} \), \( p_{12}^{fb} = \frac{\pi^n}{n \pi} \)

**Proof.** The proof borrows those of proposition 2 and is omitted. ☐

This lemma enables us to give the interpretation of reduced form probabilities as feasible mechanism. Following lemma 2 of Armstrong (2000), we consider a seller facing \( n - 1 \) rival sellers; each opponent has probability of \( p \geq 0 \) of losing to the seller and probability of \( q > 0 \) of drawing with the seller. Then, the overall probability that the seller wins is:

\[
\frac{1}{n} \left( \frac{p + q}{n} \right)^n - p^n
\]

Hence, e.g., if two optimal probabilities are equal, that means that the public buyer needs to fairly and randomly select the winner between the associated type firms if there exist no preferred types. If two optimal probabilities are distinct, then the preferred type(s) win(s) for sure against less preferred one. These win for sure if no preferred types exist and if the optimal probabilities associated to less preferred type are different. Therefore, the
optimal selection rules in complete information framework imply that the optimal mechanisms fairly and randomly select for sure a seller in the set of types associated to the highest surplus, if such a seller exists, else in the set of types associated to the second highest surplus, and so on... Clearly, and intuitively, the first best solutions never bunch any types. Whatever the SSO, first-best efficiency implies that a procurement auction is optimal, so that, a higher expected probability is associated to a higher social surplus. The preferred type is always selected.

3.3 The second-best solution

When information becomes incomplete, the public buyer must now compare the virtual surplus \( W_{ij} \). But, contrarily to the complete information case, incentives constraints also require to satisfy (5). Following similar development as in the previous paragraph, we can exhibit three buyer surplus orderings (BSO), taking now information rents into account. We have:

- BSO 1: \( W_{11} > W_{21} > W_{12} > W_{22} \), if \( \Delta S < 2\Delta c \). Hence, the quality is never profitable for the buyer.

- BSO 2: \( W_{21} > W_{11} > W_{22} > W_{12} \), if \( 2\Delta c < \Delta S < 6\Delta c \). Thus, the quality is desirable whichever the efficiency, but not between the highest and the lowest costs.

- BSO 3: \( W_{21} > W_{22} > W_{11} > W_{12} \), if \( \Delta S > 6\Delta c \). Here, the quality is always profitable for the buyer.

The following proposition presents the different selection rules.

**Proposition 2** With incomplete information, the selection rules are:

- \( p_{11} = \frac{1 - (1 - \pi)^n}{n\pi} \), \( p_{21} = \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n\pi} \), \( p_{12} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n\pi} \), \( p_{22} = \frac{\pi^n}{n\pi} \), within BSO 1;

- \( p_{21} = p_{11} = \frac{1 - (1 - 2\pi)^n}{2n\pi} \), \( p_{22} = p_{12} = \frac{(1 - 2\pi)^n}{2n\pi} \), within BSO 2;

- \( p_{11} = p_{12} = p_{21} = p_{22} = \frac{1}{n} \), within BSO 3.

**Proof.** See Appendix B. ⊓⊔

With incomplete information, procurement auction is optimal only for a single BSO, namely BSO 1. In all but the first situation, bunching occurs. For BSO 3, the take-it-or-leave-it is optimal. Otherwise, a mixed procedure becomes optimal that characterizes the situations when seeking bids and simultaneously randomly select an individual supplier among some submitted bids are optimal. If BSO subset corresponds to incentives ordering subset, the optimal mechanism implies separation; otherwise each firm has the same probability of winning the contract, whatever the contract they offer. So, the
optimal selection rule does not distinguish between firms in the subset. The latter case arises because the gross surplus of the completion of the contract is increasing in the quality level, but incentives tend to discriminate against high cost firms, i.e. favoring low quality firms. Buyer's preferences and incentives work in an opposite way. Hence, except in BSO 1, a procurement auction cannot be exploited despite the presence of many potential sellers. Nevertheless, the multidimensionality of private information ensures the benefit of partial separation since the efficiency parameter limits the conflict between incentive ordering and quality valuation, so that a take-it-or-leave-it is not optimal, except for BSO 3. It follows that procurement auction (in BSO 1) or take-it-or-leave-it (in BSO 3) trading mechanisms only appear to be polar cases of the mixed procedure. Basic trading institutions are so generally under-optimal. Indeed, in the procurement literature, these basic procedures are based on models that assume monodimensional asymmetric information and monotonicity of virtual costs implying the monotonicity of buyer's preferences. Our more realistic multidimensional approach breaks down the latter monotonicity, because a higher cost may reflect either a higher quality or a lower efficiency. Therefore, except for polar cases, the general solution is a mixed procedure that combines traditional procurement mechanisms. In Myerson (1981) or Che (1993), private information only rely on efficiency and so under regularity assumptions, separation of types (auction like mechanism) is optimal. In Manelli and Vincent (1995), quality is the only private information, and may increases costs so that take-it-or-leave-it is the optimal procedure, bunching occurs under regularity assumptions.

3.4 The distortions: weak or strong inefficiency

In this subsection, we focus on allocative efficiency according to the nature of information rents. Recall that the preference of the buyer with incomplete information (the BSO) are distorted away from the complete information preferences (SSO) because information rents must be added to costs.

As for example, recall that a 22-type is preferred to a 12-type with incomplete information if $\Delta S > 2\Delta c$, while the former is preferred to the latter with complete information if $\Delta S > \Delta c$. Besides, incentives require to put probability for 12-type at least equal to the one associated to a 22-type. Hence, if $\Delta S > 2\Delta c$, the seller, with incomplete information, prefers a 22-type but he cannot do better than putting the probabilities equal (i.e. pooling types). Conversely, if $\Delta c < \Delta S < 2\Delta c$, the seller with incomplete information prefers a 12-type and can put a higher probability to this latter (i.e. separating types) while it does not correspond to its complete information first best. These forms of inefficiency are closely related to the common value aspect of our framework (i.e. quality directly affects the gross surplus).

Following Morand and Thomas (2003), we will refer to neutral rents if the buyer's preferences correspond to social preferences. Otherwise, they
are said non-neutral. Formally, with neutral rents, the buyer’s preferences with incomplete information correspond to the complete information ones. This implies that SSO $w_{ij} > w_{i'j} > w_{i'j'}$ corresponds to the BSO $W_{ij} > W_{i'j} > W_{i'j'}$ (hereafter $ij \succ i'j \succ i'j'$). To begin, let us define two kinds of inefficiency.

**Definition 3** Let a procurement auction with complete information leading to $p_{ij}^{fb} > p_{i'j}^{fb}$. That is, when observing characteristics, the buyer selects for sure a seller $ij$ against a seller $i'j'$. Then, an allocation with incomplete information is said:

- **weakly inefficient if the buyer puts the same probability to both sellers** ($p_{ij} = p_{i'j'}$). That is, if the buyer randomly selects the winner amongst both types of sellers.
- **strongly inefficient if the buyer puts higher probability on $i'j'$ than $ij$** ($p_{i'j'} > p_{ij}$). That is, if the buyer selects for sure a seller $i'j'$ against a seller $ij$.

Using the relative values of $\Delta S$ and $\Delta c$, we identify five situations.

a) If $\Delta S < \Delta c$, SSO and BSO give $11 \succ 21 \succ 12 \succ 22$.

BSO and SSO correspond, so rents are neutral. From proposition 1 and 2, the optimal procurement auctions lead to the same expected probability for each type $ij$, $p_{ij}^{fb} = p_{ij}$. The incomplete information procurement auction is then efficient since the same seller than with complete information is selected.

b) If $\Delta c < \Delta S < 2\Delta c$, SSO is $21 \succ 11 \succ 22 \succ 12$ while BSO is $11 \succ 21 \succ 12 \succ 22$.

BSO and SSO do not correspond. Hence rents are non-neutral. In this case, proposition 1 and 2 show that the optimal mechanism is a procurement auction in both informational settings. But they differ since we have $p_{11}^{fb} > p_{21}^{fb}$ and $p_{22}^{fb} > p_{12}^{fb}$ with complete information and $p_{11} > p_{21}$ and $p_{12} > p_{22}$ with incomplete information. The optimal mechanism is then strongly inefficient.

c) If $2\Delta c < \Delta S < 3\Delta c$, SSO and BSO give $21 \succ 11 \succ 22 \succ 12$.

BSO and SSO correspond and rents are neutral. However, from proposition 2 the optimal mechanism is a mixed procedure generating bunching of types 21-11 and 22-12. It leads to allocative inefficiency because type 11 (respectively 12) has the same expected probability to be selected than 21 (respectively 22) even if it is strictly less preferred by the buyer. But, inefficiency remains weak: it does not preclude selecting the same seller than with complete information.

d) If $3\Delta c < \Delta S < 6\Delta c$, SSO is $21 \succ 22 \succ 11 \succ 12$ while BSO is $21 \succ 11 \succ 22 \succ 12$.

Hence, rents are non-neutral because SSO and BSO do not correspond. With complete information, we have $22 \succ 11$ while incomplete information
yields $11 > 22$. We have not only strong inefficiency because we have $p_{22}^{fb} > p_{11}^{fb}$ and $p_{11} > p_{22}$, but also weak inefficiency since $p_{21}^{fb} > p_{11}^{fb}$ and $p_{21} = p_{11}$ and $p_{22}^{fb} > p_{12}^{fb}$ and $p_{22} = p_{12}$.

e) If $\Delta S > 6\Delta c$, SSO and BSO give $21 > 22 > 11 > 12$.

From proposition 2, all types are bunched. Despite a take-it-or-leave-it offer, inefficiency is weak because the preferred type can be selected, even if he has the same expected probability to be selected than the other types. This result is obtained because rents are neutral, so that SSO and BSO correspond. All these results can be summarized in the following proposition:

**Proposition 4** The optimal procurement institution is:

- efficient in case a,
- weakly inefficient in cases c and e,
- strongly inefficient in cases b and d.

When quality is intrinsic, the optimal procurement is thus efficient only in case a. Otherwise, it is inefficient. With neutral rents (cases c and e), a more bunching mechanism is less efficient. But inefficiency remains weak. On the other hand, if rents are non-neutral (cases b and d), then inefficiency may be strong, and a more separating mechanism does not necessarily enhance efficiency. This proposition may seem rather counter-intuitive because it may oppose allocative efficiency and separation. Actually, non-neutral rents may lead the optimal selection rules to select a seller who would not have been selected with complete information. This result departs from traditional auction theory where inefficiency may appear due to some non-regularity, but remains weak. It is a direct effect of the non-responsiveness which may appear in our common value setting. Such strong inefficiency (which may also appear in other screening models with common value) is particularly striking here because it directly affects the selection rules and leads the seller to select for sure a less preferred type from a first best point of view.

### 3.5 Implementation

What still has to be done is to detail the payment rule and to propose an implementation of the optimal mechanisms. Given firm rents, we can derive the associated optimal payment:

$$T_{ij} = R_{ij} + p_{ij}c_{ij}$$

Let $t_{ij}^*$ the ex-post payment allowed to a selected firm with quality $i$ and efficiency $j$. With $p_{ij}^*$ the optimal probabilities given in appendix B, we

---

9 We obtain a result close to the optimality of bunch in optimal asymmetric auctions (see Myerson 1981) even with symmetric bidders.
obtain:

\[ t_{22}^* = c_{22} \]

\[ t_{12}^* = c_{12} - \Delta c \frac{p_{22}^*}{p_{12}^*} \]

\[ t_{21}^* = c_{21} - \Delta c \left( \frac{p_{22}^*}{p_{21}^*} + \frac{p_{12}^*}{p_{21}^*} \right) \]

\[ t_{11}^* = c_{11} - \Delta c \left( \frac{p_{22}^*}{p_{11}^*} + \frac{p_{12}^*}{p_{11}^*} + \frac{p_{21}^*}{p_{11}^*} \right) \]

Note that if the selection rule pools a subset of types, the corresponding payments are equal, and correspond to the highest cost in this subset. We briefly present here an implementation of the optimal solution, even if it is not the only one. The following shows that specific scoring mechanisms with strictly less scores than types can be used. Note before that the implementation must incorporate two features: the sellers must reveal two private parameters and the selection rule must distinguish as many subsets of tenders as distinct optimal probabilities. The following scoring rule implements the optimal mechanisms:

**Proposition 5** Firms announce a bi-dimensional tender \((q_i, \theta_j)\). The optimal scoring rule affects value \(\sigma_{ij}\) to tender \((q_i, \theta_j)\), with \(\sigma_{ij} \in \{1, 2, 3, 4\}\).

**Buyer’s Surplus Ordering** \(\sigma_{11} \quad \sigma_{21} \quad \sigma_{12} \quad \sigma_{22}\)

\[
\begin{align*}
W_{11} & > W_{21} > W_{12} > W_{22} & 4 & 3 & 2 & 1 \\
W_{21} & > W_{11} > W_{22} > W_{12} & 3 & 3 & 1 & 1 \\
W_{21} & > W_{22} > W_{11} > W_{12} & 1 & 1 & 1 & 1
\end{align*}
\]

The firm matching the highest score wins the contract. If at least two firms match the same score, and if no other firm has a higher score, they are fairly and randomly selected. A \(ij\)-type winning firm obtains payment \(t_{ij}^*\).

In the first case, the mechanism assigns highest scores to lowest costs. In the second case the highest score is given to both lowest costs, but the mechanism does not distinguish between high and low quality firms while in the last case, each firm obtains the same score. Roughly speaking, in order to reflect the optimal trade-off between incentive compatibility and buyer preferences, the optimal scoring rules distort costs so that \(C_{ij} > C_{i'j'}\) does not necessarily implies \(\sigma_{ij} < \sigma_{i'j'}\). Note that the traditional trading institution (auction and take-it-or-leave-it) can be interpreted as special cases of our generic procedure. Actually, an auction corresponds to the polar case where the number of scores corresponds to the number of types while totally non-competing mechanism correspond to a simple scoring rule with only one score \(\sigma_{ij} = 1 \forall i, j = 1, 2\). The discrete approach does not enable us to derive more concrete rules. Nevertheless, it deserves some comments. As noted in the introduction, the RFQ practice generally involves scoring rules
too. These scoring rules may, or may not, as we obtain, be identical to the auctioneer’s true preferences. But nevertheless any quote obtains different score. In our setting, two or more different quotes may obtain the same score, and this does not correspond to any actual procurement practices.

4 Extension

In this section, we tackle with the alternative cost ordering. In this second case, it is always less costly for a seller to offer a low quality. Then, we have:

\[ c_{12} < c_{21} \]

It implies that costs are ordered as:

\[ c_{11} < c_{12} < c_{21} < c_{22} \]

Combining incentives constraints lead to:

\[ p_{11}^e \geq p_{12}^e \geq p_{21}^e \geq p_{22}^e \]  \( (7) \)

where the superscript \( e \) denotes the case studied in this extension. This suggests that the three binding incentive constraints are now such that a 11-type must not be tempted to choose the 12-type contract, a 12-type to choose 21-type, a 21-type to choose 22-type.\(^{10}\) Besides, let us assume the simplifications of the preceding section on cost increases and on the prior. Since \( R_{22}^e = 0 \) still ensures participation of all types, informational rents can be rewritten as:

\[ R_{22}^e = 0 \]
\[ R_{21}^e = \Delta c \frac{p_{22}^e}{p_{22}} \]
\[ R_{12}^e = \Delta c (p_{22}^e + p_{21}^e) \]
\[ R_{11}^e = \Delta c (p_{22}^e + p_{21}^e + p_{12}^e) \]

So the objective function becomes after collecting terms:

\[ E(W^e) = n\pi \{p_{11}^e (S_1 - c_{11}) + p_{12}^e(S_1 - (c_{12} + \Delta c)) + p_{21}^e(S_2 - (c_{21} + 2\Delta c)) + p_{22}^e(S_2 - (c_{22} + 3\Delta c)) \} \]
\[ = n\pi \{p_{11}^e (S_1 - C_{11}^e) + p_{12}^e(S_1 - C_{12}^e) + p_{21}^e(S_2 - C_{21}^e) + p_{22}^e(S_2 - C_{22}^e) \} \]
\[ = n\pi \sum_{i,j} p_{ij}^e W_{ij}^e \]

\(^{10}\) So in appendix A, constraints (8), (13) and (16) bind.
where $C_{ij}^e$ and $W_{ij}^e$ are respectively the virtual cost and the virtual surplus of type $ij$ in the extension. The major difference with respect to the preceding case is that virtual cost of type 12 is $c_{12} + \Delta c$ instead of $c_{12} + 2\Delta c$, whereas those of type 21 is $c_{21} + 2\Delta c$ instead of $c_{21} + \Delta c$. Now, a 12-type performs better than a 21-type in terms of virtual cost, whereas the reverse is true in terms of quality. In the preceding case, a 21-type was always dominating a 12-type in terms of virtual cost and quality. Finally, the relaxed problem becomes:

$$\max_{p_{ij}^e} E(W^e) = n\pi \left[ \sum_{i,j} p_{ij}^e W_{ij}^e \right] \quad s.t. \ (4), (7)$$

Solving the problem follows the same lines that in the preceding case. So, we only stress the main steps and we leave it to the reader to check the detail of the calculus.

### 4.1 The optimal procurements

We begin by expositing the complete information solution. Four SSO can be exhibited:

- **Quality is never socially desirable** if $\Delta S < \Delta c$, so that SSO$^e 1$ is $w_{11} > w_{12} > w_{21} > w_{22}$.
- **Quality is socially desirable only for intermediate costs** if $\Delta c < \Delta S < 2\Delta c$, then SSO$^e 2$ is $w_{11} > w_{21} > w_{12} > w_{22}$.
- **Quality is socially desirable whatever the efficiency**, but not between the highest and lowest costs if $2\Delta c < \Delta S < 3\Delta c$, so that SSO$^e 3$ is $w_{21} > w_{11} > w_{22} > w_{12}$.
- **Quality is always socially desirable** if $\Delta S > 3\Delta c$, then SSO$^e 4$ is $w_{21} > w_{22} > w_{11} > w_{12}$.

Thus we obtain four first-best optimal mechanisms:

**Proposition 6** The first-best optimal procurement selection rules are:

- **within SSO$^e 1$**: $p_{11}^{fb^e} = \frac{1 - (1 - \pi)^n}{n\pi}$, $p_{12}^{fb^e} = \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n\pi}$,
  
  $p_{21}^{fb^e} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n\pi}$,
  
  $p_{22}^{fb^e} = \frac{\pi^n}{n}$;

- **within SSO$^e 2$**: $p_{11}^{fb^e} = \frac{1 - (1 - \pi)^n}{n\pi}$, $p_{21}^{fb^e} = \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n\pi}$,
  
  $p_{12}^{fb^e} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n\pi}$,
  
  $p_{22}^{fb^e} = \frac{\pi^n}{n}$;

- **within SSO$^e 3$**: $p_{21}^{fb^e} = \frac{1 - (1 - \pi)^n}{n\pi}$, $p_{11}^{fb^e} = \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n\pi}$,
  
  $p_{22}^{fb^e} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n\pi}$,
  
  $p_{12}^{fb^e} = \frac{\pi^n}{n}$;
\[ p_{11}^{f_{b}} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n\pi}, \quad p_{12}^{f_{b}} = \frac{\pi^n}{n} \]

Whatever the SSO\(e\), first-best efficiency implies again that a procurement auction is optimal, so that, a higher expected probability is associated to a higher social surplus. So, the preferred type is always selected. When information becomes incomplete, BSO are such that:

- If \(\Delta S < 2\Delta c\), quality is never desirable for the buyer and BSO\(e\) 1 is \(W_{11} > W_{12} > W_{12} > W_{22}\).

- If \(2\Delta c < \Delta S < 4\Delta c\), quality is desirable for the buyer only for intermediate costs and BSO\(e\) 2 is \(W_{11} > W_{21} > W_{12} > W_{22}\).

- If \(4\Delta c < \Delta S < 5\Delta c\), quality is desirable for the buyer only for efficient types and BSO\(e\) 3 is \(W_{21} > W_{11} > W_{12} > W_{12}\).

- If \(5\Delta c < \Delta S < 6\Delta c\), quality is desirable for the buyer whatever the efficiency, but not between the highest and lowest costs and BSO\(e\) 4 is \(W_{21} > W_{11} > W_{22} > W_{12}\).

- If \(\Delta S > 6\Delta c\), quality is always desirable for the buyer and BSO\(e\) 5 is \(W_{21} > W_{22} > W_{11} > W_{12}\).

So, we can formulate the following proposition:

**Proposition 7** With incomplete information, the optimal procurement selection rules are within BSO\(e\):

1. \(p_{11}^{e} = \frac{1 - (1 - \pi)^n}{n\pi}, \quad p_{12}^{e} = \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n\pi}\),
\(p_{21}^{e} = \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n\pi}, \quad p_{22}^{e} = \frac{\pi^n}{n}\)

2. \(p_{11}^{e} = \frac{1 - (1 - \pi)^n}{n\pi}, \quad p_{12}^{e} = p_{21}^{e} = \frac{(1 - \pi)^n - (1 - 3\pi)^n}{n\pi}, \quad p_{22}^{e} = \frac{\pi^n}{n}\)

3. \(p_{11}^{e} = p_{12}^{e} = p_{21}^{e} = \frac{1 - (1 - 3\pi)^n}{n\pi}, \quad p_{22}^{e} = \frac{\pi^n}{n}\)

4. \(p_{11}^{e} = p_{12}^{e} = p_{21}^{e} = p_{22}^{e} = \frac{1}{n}\)

5. \(p_{11}^{e} = p_{12}^{e} = p_{21}^{e} = p_{22}^{e} = \frac{1}{n}\)

As in the preceding case, we can observe that the generic trade institution is a mixed procedure that partially pools types and partially separates others (BSO\(e\) 2 and 3). Thus auction (BSO\(e\) 1) and take-it-or-leave-it mechanisms (BSO\(e\) 4 and 5) appear again as polar trading institutions. In this extension, we note that mixed procedure may appear more frequently than in the preceding case. It comes from the fact that nowadays, a 21-type which is preferred to a 12-type by the buyer in terms of quality is no more preferred in terms of virtual cost. This situation creates more conflicts between the
buyer's incentives and private incentives. Mixed procedures appear under a wider range of situations.

4.2 Efficiency of the optimal mechanism

As before, it can be easily shown that for a certain range of parameters, SSO* and BSO* may differ. It has an important implication on allocative efficiency. More precisely, if we compare the relative values of $\Delta S$ and $\Delta c$ multiples, the extension leads to seven distinct subcases. We resume these different alternatives in the following proposition.

Proposition 8 The optimal mechanism is:

- efficient if $\Delta S < \Delta c$,
- strongly inefficient via intermediate costs if $\Delta c < \Delta S < 2\Delta c$,
- strongly inefficient via the lowest cost and weakly inefficient via intermediate costs if $2\Delta c < \Delta S < 3\Delta c$,
- strongly inefficient via the lowest and the highest costs and weakly inefficient via intermediate costs if $3\Delta c < \Delta S < 4\Delta c$,
- strongly inefficient via the highest cost and weakly inefficient via the three lowest costs if $4\Delta c < \Delta S < 5\Delta c$,
- weakly inefficient if $5\Delta c < \Delta S < 6\Delta c$,
- weakly inefficient if $6\Delta c < \Delta S$.

In this case, an optimal mechanism can be both weakly and strongly inefficient. It is new in regard to the preceding section. Once again, this is due to the more conflictual aspect of the buyer's incentives with the private incentives of this case.

5 Conclusion

We have modeled the optimal multidimensional procurement when quality is non-contractible. Contrary to the traditional literature, we drop the monodimensionality hypothesis and design buyer-optimal mechanisms when sellers' private characteristics differ in many dimensions, quality and efficiency. Assuming that quality is costly to produce that high quality contracts are more costly than low quality ones, we derive the optimal procurement mechanisms in a variety of situations, depending on the relative profitability of a quality increase. We show that most of these procedures may imply some partial pooling of types, which is very far from actual practice or public procurement legislations, which either imply a totally separating mechanism (auction) or a totally pooling mechanism (in practice, non contractible attributes are simply not used in the procurement mechanisms). We distinguish bunch resulting from socially neutral rents to distorting ones. The first
one generates only weak inefficiency while the second may generate strong inefficiency. In this case, and contrary to the standard results in mechanism design, a more separating mechanism may be a less efficient one. A natural extension of our model would imply tackling with continuous distribution of types even if such a generalization seems to be very hard to complete.

6 Appendix

Appendix A

The 12 incentive constraints are:

\[ R_{11} \geq R_{12} - p_{12}(c_{11} - c_{12}) \] \hspace{1cm} (8)
\[ R_{11} \geq R_{22} - p_{22}(c_{11} - c_{22}) \] \hspace{1cm} (9)
\[ R_{11} \geq R_{21} - p_{21}(c_{11} - c_{21}) \] \hspace{1cm} (10)
\[ R_{12} \geq R_{11} - p_{11}(c_{12} - c_{11}) \] \hspace{1cm} (11)
\[ R_{12} \geq R_{22} - p_{22}(c_{12} - c_{22}) \] \hspace{1cm} (12)
\[ R_{12} \geq R_{21} - p_{21}(c_{12} - c_{21}) \] \hspace{1cm} (13)
\[ R_{21} \geq R_{11} - p_{11}(c_{21} - c_{11}) \] \hspace{1cm} (14)
\[ R_{21} \geq R_{12} - p_{12}(c_{21} - c_{12}) \] \hspace{1cm} (15)
\[ R_{21} \geq R_{22} - p_{22}(c_{21} - c_{22}) \] \hspace{1cm} (16)
\[ R_{22} \geq R_{11} - p_{11}(c_{22} - c_{11}) \] \hspace{1cm} (17)
\[ R_{22} \geq R_{12} - p_{12}(c_{22} - c_{12}) \] \hspace{1cm} (18)
\[ R_{22} \geq R_{21} - p_{21}(c_{22} - c_{21}) \] \hspace{1cm} (19)

Manipulating these inequalities yields the necessary probabilities ordering. Actually, inequalities (8) and (11) yield:

\[ p_{11}(c_{12} - c_{11}) \geq R_{11} - R_{12} \geq p_{12}(c_{12} - c_{11}) \]

but \( c_{12} - c_{11} > 0 \) so:

\[ p_{11} \geq p_{12}. \]

Proceeding in the same way pair by pair, the set of second order incentive constraints are summarized by equation (5).
Appendix B Proof of Proposition 2

Solving the relaxed problem depends upon the relative weight of virtual surplus associated to each type \( ij \). The buyer’s preferences directly represent a particular ordering of virtual surplus. Let us define:

\[
W^1 = \max_{i,j} W_{ij} \\
W^2 = \max_{i,j} \left[ W_{ij} \setminus W^1 \right] \\
W^4 = \min_{i,j} W_{ij} \\
W^3 = \min_{i,j} \left[ W_{ij} \setminus W^4 \right]
\]

Hence, the buyer’s preferences are:

\[
W^1 > W^2 > W^3 > W^4 \quad (20)
\]

Let \( p^s \) be the reduced form probability to select a firm generating a virtual surplus \( W^s, \ s = 1, \ldots, 4 \). For a given quality, the buyer still prefers high efficiency. Thus, we have:

\[
W_{11} > W_{12} \\
W_{21} > W_{22} \quad (21)
\]

Moreover, we get:

\[
W_{21} > W_{12} \quad (22)
\]

since \( S_2 > S_1 \) and \( C_{21} < C_{12} \). The general formulation of the restricted (i.e. taking into account part of the feasibility constraints) relaxed problem is so:

\[
\begin{aligned}
\max_{p^s} E(W) &= n \sum_{s=1}^{4} \pi p^s W^s \\
\text{s.t. } n \pi p^1 &\leq 1 - (1 - \pi)^n \quad (FC_1) \\
n \pi \sum_{s=1}^{2} p^s &\leq 1 - (1 - 2\pi)^n \quad (FC_2) \\
\sum_{s=1}^{3} p^s &\leq 1 - (1 - 3\pi)^n \quad (FC_3) \\
\sum_{s=1}^{4} p^s &= 1 \quad (FC_4) \\
\text{(5), (21) and (22)}
\end{aligned}
\]

We consider only a subset of feasibility constraints (4 out of 15) because the buyer’s preferences suggest this subset. We will show that whatever
the buyer’s preference ordering is, the optimal mechanism is a particular solution of this general formulation. The neglected constraints are checked \textit{ex post} (see Appendix C for feasibility constraints). Our demonstration involves three successive steps. In the first step, we establish the properties of the inequalities system \((FC_1)\) \((FC_2)\) \((FC_3)\) and \((FC_4)\). In the second step, we solve \((P'')\) which is equivalent to \((P')\) without \((5), (21)\) and \((22)\). In the last step, we resolve \((P')\) by confronting \((5)\) to the solution of \((P'')\), given \((21)\) and \((22)\).

**Step 1. Property of the inequalities system** First, the following lemma is needed.

**Lemma 9** Let \(f(x, y) = \frac{(1-x)^n-(1-y)^n}{n(y-x)}\) with \(\begin{cases} x \geq 0 \\ 1 \geq y \\ y > x \end{cases}\) then:

\[
\frac{df(.)}{dx} < 0 \text{ if } dx > 0 \text{ and } dy > 0
\]

**Proof.** We have \(f_x(x, y) = \frac{(1-x)^n-(1-y)^n-n(1-x)^{n-1}(y-x)}{n(y-x)^2}\). Let \(g(x, y) = (1-x)^n-(1-y)^n-n(1-x)^{n-1}(y-x)\). Clearly, we have: \(g(y, y) = 0\) and \(g_y(x, y) = n(1-y)^{n-1} - n(1-x)^{n-1} = n[(1-y)^{n-1} - (1-x)^{n-1}] < 0\) because \(y > x\). But, from \(g(x, y) = 0\) if \(x = y\), \(y\) only may be increased, so \(g(x, y) < 0\). Hence the \(x\) increase can be obtained only if \(x < y\) with \(g(y, y) = 0\). Therefore: \(g(x, y) < 0\) \(\forall x, \forall y\), and \(f_x(x, y)\) is so. Similarly with \(f_y(x, y) = \frac{-(1-x)^n+(1-y)^n+n(1-y)^{n-1}(y-x)}{n(y-x)^2}\), we define \(h(x, y) = -(1-x)^n + (1-y)^n + n(1-y)^{n-1}(y-x)\). Following similar developments, we can check that \(h(x, y) < 0\) \(\forall x, \forall y\) and therefore \(f_y(x, y) < 0\). It follows:

\[
\frac{df(x, y)}{dx} = f_x(x, y)dx + f_y(x, y)dy < 0 \text{ if } dx \text{ and } dy > 0
\]

(23)

Hence, when \(x\) and \(y\) jointly increase, \(f(.)\) decreases. \(\square\)

**Property 1:** If constraints \((FC_1)\) to \((FC_4)\) bind, then \(p^1 > p^2 > p^3 > p^4\).

**Proof.** If constraints bind, we obtain:

\[
\begin{align*}
p^1 &= \frac{1 - (1 - \pi)^n}{n\pi} = f(0, \pi^1) \\
p^2 &= \frac{(1 - \pi)^n - (1 - 2\pi)^n}{n\pi} = f(\pi, 2\pi) \\
p^3 &= \frac{(1 - 2\pi)^n - (1 - 3\pi)^n}{n\pi} = f(2\pi, 3\pi) \\
p^4 &= \frac{(1 - 3\pi)^n}{n\pi} = f(3\pi, 1)
\end{align*}
\]
Hence, \( p^1 > p^2 > p^3 > p^4 \) from (23). \( \Box \)

This property shows that if the feasibility constraints bind, then all the probabilities are distinct. Hence, if two probabilities are equal then we must have at least one feasibility constraint slack. The following property determines which one.

**Property 2:** If probabilities \( p^s \) to \( p^t \) are equal with \( s \in \{1, 2, 3\} \), \( t \in \{2, 3, 4\} \), \( s < t \) and constraints \((FC_{s-1})\) (if it exists) and \((FC_t)\) bind, then constraints \((FC_s)\) to \((FC_{t-1})\) are slack.

**Proof.** (A) If, \( p^2 = p^1 \) and \((FC_2)\) binds, then \( p^1 = p^2 = \frac{1-(1-2\pi)^n}{2n\pi} = f(0, 2\pi) \). But, if \((FC_1)\) binds, \( p^1 = \frac{1-(1-\pi)^n}{n\pi} = f(0, \pi) > f(0, 2\pi) = p^1 \), which implies a contradiction, so \((FC_1)\) is slack.

(B) If \( p^3 = p^2 \) and \((FC_3)\) and \((FC_1)\) are binding, then \( p^2 = p^3 = \frac{(1-\pi)^n-(1-3\pi)^n}{2n\pi} = f(\pi, 3\pi) \). But, if \((FC_2)\) binds, \( p^2 = f(\pi, 2\pi) > f(\pi, 3\pi) = p^2 \), which implies a contradiction, so \((FC_2)\) slack.

(C) If \( p^3 = p^2 = p^1 \) and \((FC_3)\) binds, then \( p^1 = p^2 = p^3 = \frac{1-(1-3\pi)^n}{3n\pi} = f(0, 3\pi) \). But, if \((FC_1)\) binds, \( p^1 = f(0, \pi) > f(0, 3\pi) = p^1 \), which implies a contradiction, so \((FC_1)\) slack. If \((FC_2)\) binds, from (A) we obtain \( p^1 = p^2 = f(0, 2\pi) > f(0, 3\pi) = p^1 = p^2 \), which implies a contradiction, so \((FC_2)\) slack.

(D) If \( p^4 = p^3 \) and \((FC_4)\) and \((FC_2)\) bind, then \( p^3 = p^4 = \frac{(1-2\pi)^n}{2n\pi} = f(2\pi, 1) \). But, if \((FC_3)\) binds, \( p^3 = f(2\pi, 3\pi) > f(2\pi, 1) \), which implies a contradiction, so \((FC_3)\) is slack.

(E) If \( p^4 = p^3 = p^2 \) and \((FC_4)\) and \((FC_1)\) bind, then \( p^2 = p^3 = p^4 = \frac{(1-\pi)^n}{3n\pi} = f(\pi, 1) \). But, from (B), \( p^3 = p^2 \) with \((FC_3)\) and \((FC_1)\) binding, implies \((FC_2)\) not binding and \( p^2 = p^3 = f(\pi, 3\pi) > f(\pi, 1) \), which implies a contradiction, so \((FC_2)\) and \((FC_3)\) are slack.

(F) If \( p^4 = p^3 = p^2 = p^1 \) and \((FC_4)\) binds, then \( p^1 = p^2 = p^3 = p^4 = \frac{1}{n} = f(0, 1) \). But, if \((FC_3)\) binds, \( p^1 = p^2 = p^3 = f(0, 3\pi) > f(0, 1) \), which implies a contradiction, so \((FC_3)\) is slack. Similarly, if \((FC_2)\) binds \( p^1 = p^2 = f(0, 2\pi) > f(0, 1) \) which implies a contradiction, so \((FC_2)\) is slack and finally, if \((FC_1)\) binds, \( p^1 = f(0, \pi) > f(0, 1) = p^1 \) which implies a contradiction, so \((FC_1)\) is slack. \( \Box \)

**Step 2: The solution of (P'')** Let us consider the associated Lagrange function to \((P'')\):

\[
\mathcal{L} = n\pi \sum_{s=1}^{4} p^s W^s + \sum_{\delta=1}^{4} \lambda^{(\delta)} \left( 1 - (1-\delta\pi)^n - n\pi \sum_{s=1}^{\delta} p^s \right)
\]

Since the contract is assumed to be allocated for sure, the first order conditions give:

\[
W^1 - \lambda^{(1)} - \lambda^{(2)} - \lambda^{(3)} - \lambda^{(4)} = 0
\] (24)
\[ W^2 - \lambda^{(2)} - \lambda^{(3)} - \lambda^{(4)} = 0 \]  
\[ W^3 - \lambda^{(3)} - \lambda^{(4)} = 0 \]  
\[ W^4 - \lambda^{(4)} = 0 \]  

The complementary slackness conditions are:

\[ \lambda^{(\delta)} \left( 1 - (1 - \delta \pi)^n - n \pi \sum_{s=1}^{\delta} p^s \right) = 0, \quad \lambda^{(\delta)} \geq 0, \quad \forall \delta = 1, 2, 3 \]

\[ n \pi \sum_{s=1}^{4} p^s = 1, \quad \lambda^{(4)} \geq 0 \]

Equation (27) implies:

\[ \lambda^{(4)} = W^4 > 0 \]

So (26) leads to, given (20):

\[ \lambda^{(3)} = W^3 - W^4 > 0 \]

Following the same arguments, we have:

\[ \lambda^{(2)} = W^2 - W^3 > 0 \]

\[ \lambda^{(1)} = W^1 - W^2 > 0 \]

So each \((FC_s)\) binds. Therefore, following property 1, we have:

\[ p^1 > p^2 > p^3 > p^4. \]  

**Step 3: Confrontation of (5) to the solution of \((P''')\)**

We need to confront the probabilities ordering of step 2 with (5). We can therefore determine what feasibility constraints (from \(FC_1\) to \(FC_4\)) are not binding in the solution of the restricted relaxed problem. Given (21) and (22), we have three possible orderings for buyer’s preferences.

Case n° 1: Let \(W^1 = W_{11}, \ W^2 = W_{21}, \ W^3 = W_{12}, \ W^4 = W_{22}\).

From (28), it implies \(p_{11} > p_{21} > p_{12} > p_{22}\) with \((FC_1) - (FC_4)\) binding. Clearly, (5) is satisfied and the solutions of \((P')\) and \((P'')\) do not differ.

Thus by resolving the inequalities system, we get: \(p_{11} = \frac{1-(1-\pi)^n}{n \pi}, \ p_{21} = \frac{(1-2\pi)^n}{n \pi} - \frac{(1-3\pi)^n}{n \pi}, \ p_{12} = \frac{(1-2\pi)^n}{n \pi} - \frac{1-(1-3\pi)^n}{n \pi}, \ p_{22} = \frac{p_{11}^{\pi}}{n \pi}.\)

Case n° 2: Let \(W^1 = W_{21}, \ W^2 = W_{11}, \ W^3 = W_{22}, \ W^4 = W_{12}\). Following similar arguments as above, the solution of \((P')\) implies \(p_{11} = p_{21} > p_{12} = p_{22}\) with \((FC_2)\) and \((FC_4)\) binding. Thus, \(p_{21} = p_{11} = \frac{1-(1-2\pi)^n}{2n \pi}, \ p_{22} = p_{12} = \frac{(1-2\pi)^n}{2n \pi}.\)
Case $n^o$ 3: Let $W^1 = W_{21}$, $W^2 = W_{22}$, $W^3 = W_{11}$, $W^4 = W_{12}$ so $p_{11} = p_{12} = p_{21} = p_{22} = \frac{1}{n}$.

Let us verify the second order incentive conditions. Clearly, it has been assumed that the upward local incentive constraints (over cost) are binding. But, since the single-crossing property and the probabilities are monotonic (still over cost), standard proof in screening models shows that incentive compatibility is global.

Appendix C

Recall that, in the preceding section, we tackle only a subset of feasibility constraints. We now turn to consider and check the neglected constraints. Because this task is very tedious (11 constraints to be verified), we only derive the intuition of the proof. In $(P''')$, we have $W_1 > W_2 > W_3 > W_4$ and the constraints $FC_s$, $s = 1, \ldots, 4$ bind. If a constraint $FC_s$, $s = 1, \ldots, 4$ is binding, it means the $s$-type wins the contract for sure if there is no better opponent. Therefore, omitted constraints can not bind otherwise the contract would be allocated for sure to another type than $s$ while $s$ yields a greater virtual surplus. In order to verify this assertion, let us write the set of feasibility constraints:

I. Feasibility constraints including one probability.

$$n \pi p^i \leq 1 - (1 - \pi)^n \quad i = 1, \ldots, 4 \quad (I.i)$$

II. Feasibility constraints including two probabilities.

$$n \pi p^1 + n \pi p^2 \leq 1 - (1 - 2\pi)^n \quad (II.1)$$
$$n \pi p^1 + n \pi p^3 \leq 1 - (1 - 2\pi)^n \quad (II.2)$$
$$n \pi p^1 + n \pi p^4 \leq 1 - (1 - 2\pi)^n \quad (II.3)$$

III. Feasibility constraints including three probabilities.

$$n \pi (p^1 + p^2 + p^3) \leq 1 - (1 - 3\pi)^n \quad (III.1)$$
$$n \pi (p^1 + p^2 + p^4) \leq 1 - (1 - 3\pi)^n \quad (III.2)$$
$$n \pi (p^1 + p^3 + p^4) \leq 1 - (1 - 3\pi)^n \quad (III.3)$$

$$n \pi (p^2 + p^3 + p^4) \leq 1 - (1 - 3\pi)^n \quad (III.4)$$

Let $p^1 = \frac{1-(1-\pi)^n}{n\pi}$, $p^2 = \frac{(1-\pi)-(1-2\pi)^n}{n\pi}$, $p^3 = \frac{(1-2\pi)-(1-3\pi)^n}{n\pi}$, $p^4 = \frac{(1-3\pi)^n}{n\pi}$ then: $(I.1) \Rightarrow f(0, \pi) = f(0, \pi) \quad (I.2) \Rightarrow f(\pi, 2\pi) < f(0, \pi) \quad (I.3) \Rightarrow$
\[ f(2\pi, 3\pi) < f(0, \pi) \Rightarrow f(3\pi, 1) < f(0, \pi) \Rightarrow f(\pi, 2\pi) = f(\pi, 2\pi)^{II.1} \Rightarrow 1 - (1 - \pi)^n + (1 - 2\pi)^n - (1 - 3\pi)^n < 1 - (1 - 3\pi)^n \Rightarrow f(2\pi, 3\pi) < f(\pi, 2\pi) \text{ and so on.} \]

We show then that \((II.3)\) to \((II.6)\) are slack, \((III.1)\) binds and \((III.2)\) to \((III.4)\) are slack. So, the assertion is true. Now, the solution of \((P''')\) does not match the solution of \((P')\) if \((P')\) requires that \((p^t) \geq (p^s)\) with \(t > s\). Thus, the resolution of \((P')\) implies remove weight from probabilities \((p^s) - (p^{t-1})\) induced by \((P'')\) in order to increase \((p^t)\) such that \((p^s) - (p^t)\) become equal. From property 2, constraints \((FC_s)\) to \((FC_{t-1})\) become slack in \((P')\) and so in \((P)\). But, adding weight to \(p^t\) does not bind the neglected constraint of \((P)\) which include \(p^t\) and less than \(t\) variables. Actually, if this was not the case, it would mean that the contract would be awarded in priority to \(t\)-type rather than to \(s\)-type to \(t - 1\)-type. Such a solution would be optimal only if \(t\)-virtual surplus is greater, which is contrary to the initial assumption \(W_t < W_s\) since \(s < t\).

References


