Linear bonds valuation with interest rate models: Does it work?

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1 Introduction

For the last twenty years, the term structure of interest rates and its evolution have been largely studied and discussed. The term structure describes the relation among the yields of default risk free bonds of different maturities. A lot of models have been developed in a continuous-time framework in order to build the term structure of interest rates. The determination of this term structure is often viewed as a problem of valuation of zero coupon bonds and the approach is based on arbitrage or equilibrium arguments. The procedure used is very similar to the one proposed by Black & Scholes [1973] and Merton [1973] for their option pricing models. In the most classical models, the only state variable considered is the short term interest rate. A diffusion process, including parameters to estimate, is then used to represent the dynamics of this state variable. It is worth noticing that several multifactor term structure models (see for example Richard [1978], Brennan & Schwartz [1979], Langetieg [1980] or Longstaff & Schwartz [1992]) can be found in the literature but their implementation faces many difficulties such as the parameters estimation problem.

Next, using an arbitrage argument, a partial differential equation is derived. This equation, joined to a boundary condition, must be solved to obtain the price of the zero coupon bond for a given maturity. Sometimes, the problem can be solved analytically but it often needs to be numerically solved.

Despite the number of models encountered in the literature, very few empirical studies exist. A first approach, initiated by Brown &

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Dybvig [1986], estimates the (risk-adjusted) parameters of the model from cross-sections of prices. That is to say, for any single day (or week/month), they take the market price of every Treasury bill and bond and estimate the model parameters that produce the best least-squares fit of model prices to observed market prices. The Brown & Dybvig approach has also been used by Sercu & Wu [1997] on Belgian data. The second approach, which we broadly follow, focuses firstly on the ability of the different diffusion processes to capture the actual behaviour of the short-term interest rate [see Chan et al [1992], Broze, Scaillet & Zakoïan [1995] and De Winne [1995]]. Then, the parameters estimates are used to value interest rate derivative assets. In this paper, the prices of Belgian coupon bonds implied by some one-factor models are compared to the actual prices observed on the market.

The paper is organised as follows. In section 2, the pricing methodology is described. A data description is given in section 3. In section 4, the theoretical prices for the different models are compared to the actual prices. Section 5 concludes.

2 Pricing methodology

In a recent paper [De Winne [1995]], an empirical comparison showed that the diffusion processes chosen by Vasicek [1977], Brennan & Schwartz [1980], Cox, Ingersoll & Ross [1985] and Longstaff [1989] best describe the behaviour of the short term interest rate. This work used one-month interbank rates time series observed in different European countries during the period January 89 – December 92.

So, the prices of Belgian coupon bonds implied by these four models will be computed and compared to the actual prices observed on the market. Indeed, it is not very relevant to take into account other models which showed poor performances in capturing the behaviour of the short term interest rate.

In the chosen models, the dynamics of the short term interest rate is represented by a diffusion process of the form:

$$dr = \mu\big(r(t),t\big)\,dt + \sigma\big(r(t),t\big)\,dZ\,.$$

The price at time t of a discount bond maturing at time T is a function of $r(t)$:

$$P(t,T) = P\big(t,T,r(t)\big)\,.$$

At equilibrium, this price must satisfy the following partial differential equation:

$$
\begin{aligned}
\Big[\mu\big(r(t),t\big) - \lambda\big(r(t),t\big)\sigma\big(r(t),t\big)\Big]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} & \\
-P\big(r(t),t,T\big) + \frac{1}{2}\sigma^2\big(r(t),t\big)\frac{\partial^2 P}{\partial r^2} &= 0 .
\end{aligned}
$$

Moreover, the price of a discount bond at maturity must be equal to unity. So, the following boundary condition is added to the system:

$$P\big(t,t,r(t)\big) = 1 .$$

Unfortunately, this kind of system is not always easy to solve. So, an analytical solution can be found for the Vasicek [1977] and Cox, Ingersoll & Ross [1985] models. Longstaff [1989] derived a closed form solution for his model but the mathematical development used has been criticised by some other researchers. Up to now, no closed form solution has been derived for the Brennan & Schwartz [1980] model. So, the Monte Carlo method has been used in order to price the bonds in the framework of Brennan & Schwartz [1980] and Longstaff [1989].

2.1 Determination of the market price of risk

Once the parameters of the diffusion process have been estimated, one variable must still be specified in order to price the discount bonds: the market price of risk which has been denoted $\lambda(r(t),t)$ in this paper. This variable measures the extent to which investors require higher returns to compensate them for bearing the risk associated with the variable (here, the short term interest rate). Unfortunately, this market price of risk is not directly observable from market data. So, different methods are used to specify its form. First, the market price of risk can be found as the value which minimises the squared errors between the actual bond prices and the theoretical prices which depend on it. But, doing so, a specific market price of risk is determined for each model and the results are biased in favour of the model. This method is very similar to the computation of the implied volatilities given by the Black & Scholes [1973] option pricing model.

Another method consists in estimating all the parameters, including the market price of risk, in such a way that the model fits the actual term structure. This is the approach followed by Brown & Dybvig [1986] and Sercu & Wu [1997]. This method is questionable in that it does not account for the role of the interest rate dynamics. For example, the implied short rates of return obtained by Brown & Dybvig using the Cox, Ingersoll & Ross [1985] model are significantly overestimated. As for the results of Sercu & Wu, they show a significant difference between the risk-adjusted drift in the short term rate obtained from both Vasicek [1977] and Cox, Ingersoll & Ross [1985] frameworks.

In a first step, it is assumed that local expectations hypothesis ($\lambda = 0$) holds, so that the expected return on all discount bonds is the riskless rate. Of course, this approach is questionable and will affect the pricing accuracy but it allows to treat the competing models in a homogeneous way. Then, in a second step, the implicit market price of risk is computed for each model from the actual price of a frequently traded bond and is used for the pricing of the other bonds.

2.2 Solutions of Vasicek [1977] and Cox, Ingersoll & Ross [1985]

2.2.1 The Vasicek [1977] model

Assuming that the short term interest rate follows an Ornstein-Uhlenbeck process, Vasicek developed the price function of a discount bond for a constant market price of risk. Here is the partial differential equation that must be satisfied:

$$\left[\delta(\theta - r) - \sigma\lambda\right]\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP - \frac{\partial P}{\partial \tau} = 0.$$

The closed form solution obtained by Vasicek [1977] can be written as follows:

$$P(r,\tau) = \exp\left[\frac{1}{\delta}\left(1 - e^{-\delta\tau}\right)\left(R(\infty) - r\right) - \tau R(\infty) - \frac{\sigma^2}{4\delta^3}\left(1 - e^{-\delta\tau}\right)^2\right], \quad \tau \geqslant 0,$$

where

$$R(\infty) = \theta + \frac{\sigma\lambda}{\delta} - \frac{\sigma^2}{2\delta^2}\cdot$$

From these prices, the term structure of interest rates is then calculated:

$$R(r,\tau) = R(\infty) + \left(r(t) - R(\infty)\right)\frac{1}{\delta\tau}\left(1 - e^{-\delta\tau}\right) + \frac{\sigma^2}{4\delta^3\tau}\left(1 - e^{-\delta\tau}\right)^2, \quad \tau \geqslant 0.$$

The yield curves given by this formula start at the current level $r(t)$ of the spot rate for $\tau = 0$ and approach a common asymptote $R(\infty)$ as $\tau \to \infty$. This model shows different shapes for the term structure depending on the current value $r(t)$ of the spot rate.

2.2.2 The Cox, Ingersoll & Ross [1985] model

Cox, Ingersoll & Ross [1985] used the square root process to represent the behaviour of the short term interest rate:

$$dr = \delta(\theta - r)dt + \sigma\sqrt{r}dz .$$

So, the partial differential equation becomes:

$$\frac{\partial P}{\partial t} + \left[\delta(\theta - r) - \lambda r\right]\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial r^2} - rP = 0$$

The close form solution derived by Cox, Ingersoll & Ross [1985] is the following:

$$P\big(r(t),t,T\big) = A(t,T)e^{-B(t,T)r} ,$$

where

$$A(t,T) \equiv \left[\frac{2\gamma e^{[(\delta+\lambda+\gamma)(T-t)]/2}}{(\delta+\lambda+\gamma)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right]^{2\delta\theta/\sigma^2} ,$$

$$B(t,T) \equiv \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\delta+\lambda+\gamma)\left(e^{\gamma(T-t)}-1\right)+2\gamma} ,$$

$$\gamma \equiv \sqrt{(\delta+\lambda)^2 + 2\sigma^2} .$$

3 Data description

The purpose of this paper is to compare the prices of Belgian coupon bonds implied by some interest rate models to the actual prices observed on the market. Therefore, the actual prices of different linear bonds, the so-called OLOs, issued by the Belgian government have been collected. In spite of a public debt of some ten billions of Belgian francs, these linear bonds can be considered as default-free. Moreover, the amounts of linear bonds traded on the secondary market guarantee a correct price. Table1 presents some characteristics of these linear bonds and their actual prices.

It is worth noticing that the interest income resulting from Belgian government bonds is calculated assuming months of 30 days and years of 360 days. The amount to be paid by the buyer of bonds on the secondary market is calculated as a percentage of nominal par value increased by the interest accrued from the latest coupon payment date to the day before the settlement date. This amount is the actual price presented

Table 1: Some characteristics of belgian linear bonds and their actual prices

Code	Nominal rate	Coupon date	Maturity	Actual price
23945	8.25	01/06	1999	108.7269
24551	10.00	05/04	1996	114.5983
24753	10.00	02/08	2000	117.4935
24854	9.25	02/01	1998	107.1127
24955	9.50	28/02	1994	110.1981
25157	9.00	28/03	2003	115.6500
25258	9.00	27/06	2001	112.3750
25460	9.25	29/08	1997	110.2739
25763	8.50	01/10	2007	107.8729
25965	8.75	25/06	2002	111.1513
26066	9.00	30/07	1998	110.2900
26268	8.00	24/12	2012	100.9966

in the table. These prices are those observed on the 30th of December in 1992.

4 Coupon bond valuation and comparison

The first step is the parameters estimation of the diffusion process representing the behaviour of the short term interest rate. This step has been fully described in another paper (De Winne [1995]) using one-month interbank rates time series observed in Belgium during the period January 89 - December 92. A weekly periodicity has been chosen and the discretisation bias has been taken into account using the quasi indirect inference method (see Broze, Scaillet & Zakoïan [1995,1997] and De Winne [1995]). Table 2 gives the parameters which will be used for valuation.

So, all the necessary data are now available to price the linear bonds. Let us recall that all these models give the price of a discount bond of a given maturity. But, since a coupon bond can be considered as a portfolio of discount bonds, it will be easy to compute the coupon bond price as the sum of several discount bond prices.

It is worth to recall that a Monte Carlo method has been used to obtain the prices implied by the Brennan & Schwartz [1980] and Longstaff [1989] models. For the other two models, the closed form solution has been used.

Table 2: Models and parameters used for the coupon bond valuation

Model	Parameters
Vasicek [1977] $dr = \delta(\theta - r)dt + \sigma\, dZ$	$\delta = 0.0446875$ $\theta = 0.0860140$ $\sigma = 0.0015313$
Cox, Ingersoll & Ross [1985] $dr = \delta(\theta - r)dt + \sigma\sqrt{r}\, dZ$	$\delta = 0.0434375$ $\theta = 0.0942446$ $\sigma = 0.0047813$
Brennan & Schwartz [1980] $dr = \delta(\theta - r)dt + \sigma r\, dZ$	$\delta = 0.0435000$ $\theta = 0.0933908$ $\sigma = 0.0159375$
Longstaff [1989] $dr = \delta\,(\theta - \sqrt{r})\, dt + \sigma\sqrt{r}\, dz$	$\delta = 0.0259375$ $\theta = 0.3072289$ $\sigma = 0.0048125$

Table 3 gives the results of the coupon bond valuation with the average error(1) and the dispersion(2) of results.

These results show that the theoretical prices given by the competing models are very close to each other. This is not very surprising because these models are all based on the same state variable and differ only by the assumed dynamics of this variable. It is worth noticing that the longer the bond maturity is, the more important the error is. However, even for short term bonds, none of these models give a theoretical

(1) The average error (%) is measured, in tables 3 and 4, as:

$$\frac{100}{N}\sum_{i=1}^{n}\left|\frac{(P_{th,i} - P_{a,i})}{P_{a,i}}\right|$$

where $P_{th,i}$ represents the theoretical price of bond i and $P_{a,i}$ its actual price.

(2) The measure of dispersion (%), in tables 3 and 4, is:

$$\frac{100}{N}\sum_{i=1}^{n}\left[\frac{(P_{th,i} - P_{a,i})}{P_{a,i}} - \overline{ME}\right]$$

where

$$\overline{ME} = \frac{1}{n}\sum_{i=1}^{n}\frac{(P_{th,i} - P_{a,i})}{P_{a,i}}\,.$$

Table 3: Results of the coupon bond valuation

Code	Model				Actual Prices
	Vasicek	Cox, Ingersoll & Ross	Brennan & Schwartz	Longstaff	
23945	101.1701	100.6862	100.7412	100.6886	108.7269
24551	110.0074	109.8485	110.0746	110.0686	114.5983
24753	109.3500	108.6962	108.7627	108.6540	117.4935
24854	110.1457	109.8131	109.8838	109.8340	107.1127
24955	108.4100	108.3891	108.4872	108.4843	110.1981
25157	106.7822	105.8102	105.8935	105.8142	115.6500
25258	104.5040	103.7541	103.8196	103.7716	112.3750
25460	103.9482	103.6528	103.6583	103.6702	110.2739
25763	98.0797	96.6022	97.1631	96.9493	107.8729
25965	102.8809	102.0127	102.1119	101.9700	111.1513
26066	103.6814	103.2876	103.2802	103.2758	110.2900
26268	91.0765	89.1682	89.4337	89.2048	100.9966
Av. Err.	6.257	6.842	6.731	6.795	-
Disp.	3.321	3.693	3.638	3.685	-

price close to the actual one. A possible explanation for these bad results could be the choice of a market price of risk equal to zero.

So, the implicit market price of risk has been computed for every model from the actual price of the most traded bond (OLO 25965 with a trading volume of 4 183 millions of Belgian francs) on December 30, 1992. Then, the bond prices have been computed using the same methodology. Table 4 gives the *implicit* market price of risk (λ') used for each model and the theoretical bond prices obtained.

These results show that the use of a more appropriate value for (improve the accuracy of the valuation from every model but it is still unsatisfactory. Indeed, a mean relative error of more than 3 percent cannot be acceptable for any market participant. This valuation error is again more important for the longer term bonds. This is not surprising since the state variable used by the models compared is the short term interest rate. Intuitively, it seems logical that these models are better in valuing short term bonds.

Table 4
Implicit market price of risk (λ') used for each model and the theoretical bond prices obtained

Code	Model				Actual Prices
	Vasicek	Cox, Ingersoll & Ross	Brennan & Schwartz	Longstaff	
	(λ' = -2.10)	(λ' = 0.046)	(λ' = 2.850)	(λ' = 0.023)	
23945	105.6393	105.7769	105.7480	105.8218	108.7269
24551	111.4628	111.5673	111.7547	111.7792	114.5983
24753	115.4477	115.5532	115.5545	115.5335	117.4935
24854	113.1799	113.3275	113.3315	113.3869	107.1127
24955	108.6228	108.6468	108.7404	108.7382	110.1981
25157	116.1124	115.9889	116.0641	116.0724	115.6500
25258	111.5639	111.6068	111.6115	111.6605	112.3750
25460	106.6358	106.7770	106.7387	106.8113	110.2739
25763	113.1539	112.2519	112.5822	112.5918	107.8729
26066	107.3001	107.4484	107.4325	107.4567	110.2900
26268	112.2188	109.9056	110.0237	109.7990	100.9966
Av. Err.	3.414	3.075	3.103	3.071	-
Disp.	4.422	3.846	3.888	3.838	-

5 Conclusion

Our paper compares four interest rate models in their ability to value correctly a coupon bond. These models have been chosen according to their ability to capture the dynamics of the short term interest rate. For two models, a closed form solution exists and can be used for valuation. For the other models, a numerical technique (Monte Carlo) has been applied.

Our results suggest that the analysed models are all unable to give an accurate price for a coupon bond since the average error recorded is unacceptable. Indeed, the gap between the actual prices and the theoretical prices obtained from any model is about 3 percent and still larger for the longer term bonds. However, the principle of these interest rate models has not to be questioned by these deceiving results since they could be improved in some ways. First, multivariate models exist and are more likely to show better results. Indeed, the use of a long term interest rate as second state variable could lead to a more accurate pricing of long term bonds. On a theoretical point of view, many models have been developed in the perspective of obtaining a closed form solution but adding some additional state variables and complicating the diffusion process could involve better results. For example, Garch processes could replace the usual diffusion processes. These suggestions are only a small part of the possible improvements that are more likely to be brought to this field.

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